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## 1928-29.



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# XX.—ON SOME MISAPPLICATIONS OF THE LAW OF ERRORS, AND ON THE INTRINSIC ERROR IN FOCOMETRY.

By T. SMITH, *M.A.*, *F.Inst.P.*

*Received December* 2, 1927.

ABSTRACT.

In some physical measurements the uncertainty due to a single cause is not infinitesimal, and the precision of the mean of a large number of observations given by the law of errors is not then physically significant. An example is afforded in attempts to identify the position of an optical image. According to geometrical optics, this lies in a definite surface; but, owing to the physical properties of light, this surface cannot be identified experimentally. The use of a double cylindrical lens as a means of limiting more narrowly the space in which this surface lies is discussed.

WORKERS in a national standardising laboratory ever have in mind the need for high accuracy in all the measurements it undertakes. Unfortunately, the attainment of great accuracy nearly always involves the expenditure of much time, and therefore the cost of very precise work is high. In some measurements there is a natural limit to the accuracy attainable by what we may describe as a frontal attack, and this happens very commonly when light necessarily enters into the experiment. A familiar example is found in photometry where the radiant energy of a beam of light is evaluated by means of the visual sensation it excites. The accuracy attainable in such experiments is limited because the visual sensations to which two similarly constituted beams of radiant energy give rise will not differ unless the ratio of the energies of the two beams differs from unity by more than a certain finite quantity. It is often supposed that the uncertainty in the value of the quantity to be measured (in this case the radiant energy) to which all observations made under such circumstances are subject, can be reduced or even removed by taking a sufficient number of observations and applying the law of errors to them. This conclusion, however, rests upon a misapprehension. The theory on which the law of errors is based requires the uncertainty attributable to any single cause to be indefinitely small, and this condition is violated by observations of the type now considered. The use of formulæ (whether derived from the violated theory or not) for the uncertainty will only be justified if extended experience shows that we are not in actual fact led thereby to incorrect conclusions. The kind of experience needed to give an answer is not, however, of a common kind. It is gained when a new method of measurement is discovered which is free from the source of uncertainty present in previous methods, or when the range of uncertainty of a new method is small in comparison with those of the old. It has been the experience of a number of workers, when such opportunities of criticising the applicability of the law of errors have presented themselves, that the deductions which would have been made from the law, had it been legitimate to suppose it applicable, have in fact been in error by many times the amount the law would have indicated. A graphical representation will make the position clear. The value of the quantity

which it is the object of the experiment to determine may be represented by the distance of a variable point on a straight line, from a fixed origin.

Let $T$ in Fig. 1 represent the true value. Any individual observation may give a value different from $T$ owing to, let us say, systematic personal errors, accidental errors to which the law of errors is properly applicable, and to a single other cause which introduces uncertainty about which we can only say that it does not exceed the finite limits $\pm\eta$. Since in practice the total error may always be treated as a comparatively small quantity, any single observation, as a result of these several sources of error, will be represented by some point between two limiting points $A$ and $B$. We will assume the personal error to have been removed in some way we need not consider here. By multiplying the observations we may reduce the value of the probable error of the mean, in so far as this is due to infinitesimal accidental errors, to as small an amount as we wish, and we are then left with a finite region $CD$, where $CT=TD=|\eta|+|\varepsilon|$, $\varepsilon$ being some finite quantity, within which we are able to say it is probable the real value lies. The theory of errors, however, disregards the finite quantity $\eta$, and only throws some light on the magnitude likely to be attained by the (usually) smaller quantity $\varepsilon$.

When the special position of the uncertainty $\eta$ has been disregarded, it has often been inferred that the true value probably lies within a very small range $FG$ about a mean value $E$. This conclusion has been thought to be adequately supported by the convergence of a very great number of observations. When, how-

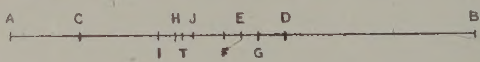


FIG. 1.

ever, a new method of measurement has been found in which the range of uncertainty $IJ$ of a single observation is very much less than $AB$, the range in the old method, it has not infrequently been discovered that the mean point $E$ reached in the older observations lies well outside the interval $IJ$, the distance $EH$ between the mid points of $FG$ and $IJ$ being several times as great as $FE$ or $EG$. Such experiences show that, under the conditions we have assumed, no reliance should be placed on a small calculated value for the probable error of the mean.

It may indeed well happen that the uncertainty in the quantity under measurement is much greater when the observations show only slight deviations from the mean than when they are considerable. This arises from the fact that in practice no reading varies from the true value by more than an amount which is well known for many types of observation. Thus, suppose the length $X$ represents the greatest distance from the real point at which settings are actually made, and let $UV$ represent the range of uncertainty about the true point $T$ due to the single cause. At ($a$) are shown five closely spaced observations, through which lines are drawn extending to the distance $X$ on either side. The only conclusion we can draw from these observations with certainty is that the real value lies between the limits $P$ and $Q$. On the other hand, from the five observations ($b$) we can conclude that the real value lies between the limits $R$ and $S$. It would ordinarily be inferred from the theory of errors that the error in the mean of set ($a$) was probably much less than that in the set ($b$). This example illustrates that the use of the law of errors, when

its fundamental assumptions are not all satisfied, may lead to quite erroneous conclusions. Clearly the most satisfactory way of attaining an accurate measurement is to increase the accuracy of a single observation, not to increase the number of observations of lower accuracy.

The kind of difficulty we have just discussed occurs when measurements depend on the accurate focussing of an image. It was laid down by Lord Rayleigh, and subsequent workers have agreed with his conclusion, that as the focus is altered the image of a point source of light hardly changes its appearance until the movement

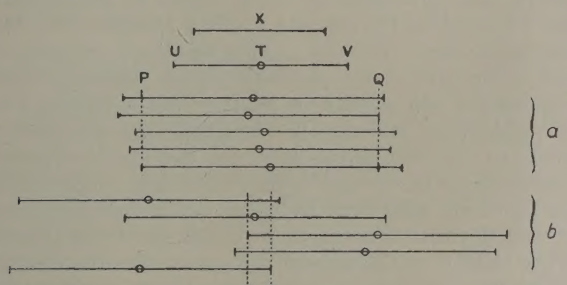


FIG. 2.

has introduced variations in the phases of the constituent ethereal wavelets exceeding one quarter of a period. Thus, physically, the focus extends through a finite space, say in the neighbourhood of the axis of a lens. Moreover, from the very nature of this criterion, it follows that the extent of this space cannot be affected by magnifying the image which is to be focussed, or like methods. On the other hand, the mathematical focus is a definite point, and it is therefore not altogether surprising that a testing laboratory should at times be pointedly requested to determine the focus within narrower limits than these natural tolerances for " perfect " focus. The difficulty may, of course, be surmounted by resorting to the Hartmann, or

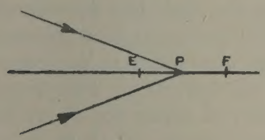


FIG. 3.

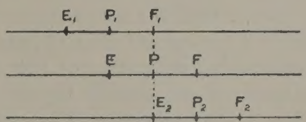


FIG. 4.

some equivalent method, of locating the focus. Usually the cost of such methods prohibits their use. It happens, however, in this, as well as in some other cases where a similar difficulty occurs, that the uncertainty of a single setting can be reduced, without altering the method of observation, by displacing the range within which good settings can be secured in both directions, leaving only a small common region within which good settings can be made. To illustrate this, let us suppose that light from a point source converges to the mathematical focus $P$. Physically, a good focus is found anywhere between $E$ and $F$. If now a positive lens is introduced

in the path of the converging light to the left of $P$ the range $EF$ will be moved towards the left; on the other hand, a negative lens will cause a movement in the opposite direction. In either case the magnitude of the displacement will depend on the power of the lens we introduce and its distance from $P$. If we have a means of producing equal displacements of both signs, and can adjust at will the magnitude of the displacement, we shall obtain the state of affairs illustrated in Fig. 4, where the mathematical focus $P$ is the only common point of the two displaced ranges $E_1F_1$ and $E_2F_2$ obtained by the use of positive and negative lenses respectively. In practice lines are used as objects instead of points, and the sharpness of the focussing is judged by the rapidity of the transition from dark to light in a direction normal to the line. The use of lines as objects enables cylindrical lenses to be interposed in the path of the converging beam without prejudice to the formation of sharp images. In particular, we may use an object consisting of two lines crossing at right angles, and employ a lens with two cylindrical surfaces having their axes parallel to the two object lines, the one surface increasing, the other reducing, the convergence of the light. If such a lens is gradually moved away from the image, a position will be reached when one or both line images will become noticeably less well defined. The plane in which the focus is examined may be adjusted until the change in the appearance occurs simultaneously in the images of both lines. A very small movement of this plane in either direction is then sufficient to introduce an unmistakable difference in the appearance of the two line images. That is to say, we are enabled by such a device greatly to reduce the intrinsic uncertainty of a single setting. We proceed to consider the theory of a lens of this kind.

It will be assumed that the distance of the double-cylindrical lens from the plane in which the mathematical image is formed will be small, so that all rays are refracted near the axis of the lens, and its aberrations may therefore be neglected. Further, it is unnecessary to consider the aberrations of the lens forming the image at $P$; we merely suppose that $P$ is the position where the range of phase variations is least, and that any inaccuracies of focus will become evident when the phase differences exceed a definite amount. We are thus virtually replacing the emergent wave converging towards $P$ as its " best " focus with an ideal spherical wave with $P$ as centre. The chief difference between the real and the ideal lens for our present discussion is that the amounts by which the cylindrical lens must be displaced from the focus to cause obvious deterioration in the image may differ in the two cases. A quantitative difference of this kind is obviously unimportant. The theoretical investigations can readily be carried out with the aid of either the characteristic function or the eikonal. The use of the most general forms for these functions, though leading readily to the interesting focal line properties of any optical instrument, is unnecessary when we intend to impose the condition that the principal axes of curvature of all the surfaces lie in the planes $y=0$ and $z=0$. It is not without interest to remark that the general method of constructing the characteristic function or the eikonal from those of its elementary constituents may fail when particular forms are assumed initially. An example is afforded by the combination of two cylinders with their axes at right angles to one another. The reasons for this failure are not obscure, but we may note that there is no failure if the formulæ established for the general case are employed to calculate the coefficients of the compound system. In the cases we shall consider the coefficients denoted by

$\mathscr{G}$ and $\mathscr{H}$ for each constituent surface are zero; it follows that in the complete system we shall have*

$$\mathscr{G}=\mathscr{H}=\mathscr{S}=\mathscr{T}=\mathscr{U}=\mathscr{V}=0,$$

so that the eikonal takes the form

$$\mathscr{E}=\frac{\mathscr{A}M^2+\mathscr{C}M'^2-2MM'}{2\mathscr{J}}+\frac{\mathscr{B}N^2+\mathscr{D}N'^2-2NN'}{2\mathscr{I}}+\ldots$$

and the characteristic function is given by

$$V=\frac{(\mathscr{C}y^2+\mathscr{A}y'^2-2yy')\mathscr{J}}{2(1-\mathscr{A}\mathscr{C})}+\frac{(\mathscr{D}z^2+\mathscr{B}z'^2-2zz')\mathscr{I}}{2(1-\mathscr{B}\mathscr{D})}+\ldots$$

As the characteristic function has been more extensively employed in problems involving the time taken by a disturbance to travel along non-stationary paths, we will illustrate the use of the eikonal in this problem. To fix ideas we will suppose that the coefficients present in $\mathscr{E}$ relate to a compound system consisting of a symmetrical lens, which is the part under test, and the two cylindrical surfaces having their axes in the planes $y=0$ and $z=0$. The surfaces of these cylinders are assumed to be at a fixed distance apart. We may denote the optical value of this interval by $2\tau$, and the powers of the two surfaces by $i$ and $j$ respectively. The power of the symmetrical lens and the magnification at which it is used are not essential factors of the problems; but, as we have included this lens in the system, we will denote its power by $\kappa$, and suppose that the object lines are infinitely distant. The direction of any particular object point is then defined by the variables $M$, $N$. Let the mid-point of the cylindrical system be optically distant $\varepsilon+1/\kappa$ from the principal point of the symmetrical system, and $\xi$ from the second eikonal reference point. Then in the two principal planes we have the following powers and distances:—

| | | | |
|---|---|---|---|
| $\kappa$ | | $\kappa$ | |
| | $\varepsilon+\frac{1}{\kappa}-\tau$ | | $\varepsilon+\frac{1}{\kappa}+\tau$ |
| $i$ | | $j$ | |
| | $\xi+\tau$ | | $\xi-\tau$ |

so that

$$\mathscr{I}=\kappa\{1-i(\varepsilon-\tau)\},$$
$$\mathscr{J}=\kappa\{1-j(\varepsilon+\tau)\},$$
$$\mathscr{C}=-\{\mathscr{J}(\xi-\tau)+\kappa(\varepsilon+\tau)\},$$
$$\mathscr{D}=-\{\mathscr{I}(\xi+\tau)+\kappa(\varepsilon-\tau)\}.$$

It follows that the terms in $\mathscr{E}$ which are affected by changes of focus are

$$-\tfrac{1}{2}\left\{(\varepsilon+\xi)(M'^2+N'^2)+\frac{j(\varepsilon+\tau)^2M'^2}{1-j(\varepsilon+\tau)}+\frac{i(\varepsilon-\tau)^2N'^2}{1-i(\varepsilon-\tau)}\right\}.$$

* For notation employed see Trans. Opt. Soc., Vol. 29, 1927-28, p. 71.

We may identify the second reference point determined by $\xi$ with the intersection of the experimentally selected image surface with the axis. The variables $M'$ and $N'$ will vary between limits determined by the pupil of the examining system, including the eye. Sometimes the symmetrical system under test will limit the radiation rather than the examining system ; but in general the facts will be represented with sufficient accuracy by assuming that the effective radiation extends over the directions given by

$$0 \leqslant M'^2+N'^2 \leqslant \alpha^2,$$

where $\alpha$ is a constant. Now the eikonal gives the value of the stationary path between that wavefront of the incident light which passes through the first reference point and the plane having the direction $M'$, $N'$ which passes through the second reference point. If, as we may, we suppose $M=N=0$, this second reference point represents the centre of an out-of-focus image, and, when the focussing error is small, light by a non-stationary route will reach this point in a time which differs negligibly from that taken to reach the plane normally by the stationary route. It follows that the variations of phase we have to consider are given by the terms of the eikonal which have been written at length above.

In the absence of the cylindrical system we insert $\varepsilon=i=j=0$, and the experimental focus may be at any point on the axis within the range determined by the condition

$$\tfrac{1}{2}\alpha^2 \mid \xi \mid \leqslant \tfrac{1}{4}\lambda,$$

that is to say, the range of uncertainty is $\lambda/\alpha^2$. When the cylindrical lens is inserted the first term of our expression represents an ordinary out-of-focus effect, due to the want of coincidence of the second reference point with the ideal geometrical focus, and the remaining two terms represent additional variations of path due to refraction at the cylindrical surfaces. These will be opposed in sign, since the signs of $i$ and $j$ differ. Any axial displacement of the cylindrical system alone is represented by a change of $\varepsilon$ without any change in the value of $\varepsilon+\xi$. If then $\tau$ is small, we can cause the phase variations due to the cylindrical refraction to assume selected values by moving this part of the system to the proper position on the axis. By altering the distance of the second reference point from the lens we make a common addition to both cylindrical effects.

Changes in the appearance of the image patch which extends in the direction $y=0$ will follow from modifications of the coefficient of $M'^2$, and similarly alterations in the patch which extends in the direction $z=0$ correspond to changes in the coefficient of $N'^2$. We know that when variations about the position of zero phase difference are made no change in appearance occurs until the variation amounts to about $\lambda/4$. In positions more remote from this intensity maximum much smaller movements of the image lead to detectable changes in the light distribution. Let us suppose that when the path variation amounts to $n\lambda$ a change $\pm\eta\lambda$ can be seen. The two focal lines will then present different appearances unless the variations of path can be represented by

$$\pm(n\pm\eta)\lambda.$$

Of the signs outside the bracket it is easy to ensure that one refers to the one line and the other to the perpendicular line. Now the actual path variations amount to

$$\tfrac{1}{2}\alpha^2\left[\varepsilon+\xi+\frac{j(\varepsilon+\tau)^2}{1-j(\varepsilon+\tau)}\right] \text{ and } \tfrac{1}{2}\alpha^2\left[\varepsilon+\xi+\frac{i(\varepsilon-\tau)^2}{1-i(\varepsilon-\tau)}\right]$$

respectively. If then

$$\frac{i(\varepsilon-\tau)^2}{1-i(\varepsilon-\tau)} \sim \frac{j(\varepsilon+\tau)^2}{1-j(\varepsilon+\tau)}=\frac{4n\lambda}{\alpha^2}$$

we shall have

$$2(\varepsilon+\xi)+\frac{i(\varepsilon-\tau)^2}{1-i(\varepsilon-\tau)}+\frac{j(\varepsilon+\tau)^2}{1-j(\varepsilon+\tau)}\leqslant\frac{4\eta\lambda}{\alpha^2}.$$

In practical applications it will be permissible to neglect the departures of the denominators from unity, and we may assume that the powers of the cylinders are equal in magnitude but opposed in sign, so that $i+j=0$. The equations then become

$$i(\varepsilon^2+\tau^2)=\frac{2n\lambda}{\alpha^2}$$

and

$$\varepsilon+\xi\leqslant\frac{2\eta\lambda}{\alpha^2}+2i\varepsilon\tau.$$

It at once follows that if the ratio $\tau/\varepsilon$ is sufficiently small (and this can obviously always be secured), we may, in the most unfavourable event, reduce the axial focussing range approximately to

$$\frac{4\eta\lambda}{\alpha^2}.$$

The criterion by which the position of the image is selected is essentially photometric. The importance of good eyesight in the observer is doubtless increased in the modified test. The position on which the eye is focussed is to be taken as the focus of the lens under test, apart from the correction due to the axial length occupied by glass in the cylindrical system. As a numerical example we may suppose $i=0{\cdot}4$D, $\tau=\frac{1}{2}$ mm., and the system under test a telescope objective of aperture $f/15$. This gives $\alpha=1/30$, and with visible light the natural sharp focussing range is about $\frac{1}{2}$ mm. The relation between $\varepsilon$ and $n$ is approximately

$$\varepsilon=50n^{\frac{1}{2}}\text{ mm.},$$

and the total focussing range is at most

$$2\left(\eta+\frac{n^{\frac{1}{2}}}{100}\right)\text{mm.}$$

Taking $n=1$, the cylindrical system is displaced five centimetres from the image, and the contribution of the second term to the focussing range is only $0{\cdot}02$ mm. If $\eta$ should prove to be as great as $1/25$ when $n=1$ (and approximate theoretical considerations lead us to anticipate a much smaller value) the focussing range would only be one-fifth as great as that in a direct examination. We are thus led to the conclusion that modifications in the experimental conditions will enable even the uncertainty attributable to the structure of light itself to be reduced to a negligible magnitude.

DISCUSSION.

Dr. L. C. Martin: I have been greatly interested in this Paper in connection with my work in ultra-violet microscopy, where the depth of focus for the ultra-violet radiation is correspondingly shorter than with visible light of longer wave-length; but the ultra-violet focus must be estimated with respect to a visual focus of some kind. The possibility of obtaining greater precision in focussing is therefore an important question; but can this be accomplished without an object of a special type, as, for example, by dividing the field and displacing the focus in one half?

Mr. Smith will be interested in a Paper published by Dr. Bevek, of Messrs. Leitz, in Germany (Marburger Sitzungsberichte 61 (1927), p. 189), in which an experimental examination of the depth of focus in the microscope is described, which was undertaken with a view to finding the relation (for the special case) between the extreme errors of setting corresponding to the visual criterion of "focus" and the "probable error" as calculated from the usual expression. The deviations from the error distribution characteristic of a pure "probability theory" case, due to the existence of the visual criterion, are clearly shown by the results. The theory of errors is not quite invariably misapplied in this connection.

Capt. C. W. Hume asked whether it was not permissible to apply the law of errors to the determination of the limits $E$, $F$, Fig. 3, of the threshold, and then to calculate the true value $P$ by assuming an appropriate relation? Would not the probable error in $P$ be thus reduced to that in $E$ and $F$?

Dr. F. E. Smith said that he had not yet read the Paper, but, prima facie, the author's conclusion seemed surprising. If two equal sources of light had to be compared photometrically it might be that no difference of illumination was visible until the ratio of one illumination to the other was 4 : 5 at one end of the threshold and 5 : 4 at the other, but the determination of these two limits depended only on the accuracy with which the scale could be set. After analysing a number of published observations, he felt that the more careful the observer the narrower would be the limits of his error.

The Author: I am glad to hear about the investigation mentioned by Dr. Martin, and hope to have an opportunity of seeing the Paper in which it is described. Capt. Hume's proposal is not free from special difficulties; the existence of the definite word "threshold" must not lead us to imagine that this abstraction corresponds to percepts equally definite. The threshold may conceivably be less definite than the region of uniform sensation. In practice the difficulty can be overcome in a more satisfactory way. In reply to Dr. F. E. Smith, it is not suggested that the mean derived from a large number of observations is necessarily, or even in a majority of measurements, in serious error; there are, however, definite instances in which it is known to have been considerably in error. If we rely upon the theory of errors, we take risks (despite all care in observations) from which we are freed by the substitution of more accurate measurements. This is in no way inconsistent with Dr. Smith's remark that the good observer obtains better results than the bad observer: in effect, this is our definition of a good observer. In the precise work contemplated in this discussion all observers employed are assumed to be good observers.

# XXI.—THE TERMINAL VELOCITY OF DROPS.

By William D. Flower, *B.Sc.*

*Received February* 1, 1928.

(*Communicated* by F. J. W. Whipple, M.A.)

Abstract.

The distance a drop of given volume has to fall through in order that it may attain a constant terminal velocity has been determined for both water and methyl salicylate. A relation connecting the distance of fall for constant velocity and the volume of the drop has been obtained.

The terminal velocities of drops of water and of methyl salicylate have been determined for drops 0·2 cm. to 0·55 cm. in diameter. A relation connecting the terminal velocity of a drop and its volume has been obtained.

THE object of these investigations was to obtain data relating to the behaviour of liquid drops falling freely through the atmosphere.

Considering the case of any one drop falling freely through the atmosphere, it is evident that after it has fallen through a certain distance the change in its velocity will be very small, and to a very near approximation the velocity will be constant. This velocity is known as the "Constant Terminal Velocity" of the drop.

No determinations appear to have been made of the distance through which a drop must fall before it attains a constant terminal velocity, but there have been various determinations of the value of the terminal velocity.

The terminal velocity of water drops has previously been determined by three investigators. (*a*) The first was Lenard,[1] who measured the velocity of an upward stream of air which would just suspend the drop. (*b*) Then Mache[2] showed that the range of the terminal velocities of rain drops agreed with Lenard's results. Mache photographed rain, and, knowing the time of exposure and the length of the streaks on the negative, he calculated the various terminal velocity values.

(*c*) Later, Schmidt[3] used two circular discs with a narrow sector cut out of each attached to the spindle of an electric motor, the speed of rotation being so arranged that the drop whose velocity was required passed through the two openings. From a knowledge of the angular velocity of the discs and their distance apart the velocity of the drop could be calculated.

## The Determination of the Distance of Fall for Constant Terminal Velocity.

The first part of the investigation consisted in determining the distance a drop of known volume has to fall in still air before it attains an approximately constant velocity.

The method employed was to liberate a drop of measured volume from a point vertically above the pan of a ballistic torsion balance, the path of a drop being increased up to a point beyond which no increase in momentum was recorded on the balance when the drop struck the pan.

EXPERIMENTAL DETAILS.

The apparatus employed was a torsion balance, shown diagrammatically (Fig. 1). The beam of the balance $AB$ was very light in construction, and carried at one end an open framework of thin glass rods, on which a filter paper was placed to catch the falling drop, a fresh piece of paper being used for each drop. To ensure that each successive piece of filter paper was placed in the same position on the framework, holes were punched in the paper through which could be placed the projections on the beam.

It was found from experience that the filter paper employed absorbed the drops with a negligible amount of splashing. In order to obtain consistent results, the filter papers used in any one experiment had to be all of the same weight. In order to check that this was the case, the necessary holes were punched in a number of papers and one placed on the framework, and the position of the counterpoise $W$ adjusted so that the scale reading of the pointer $P$ was zero. The remaining papers were then in turn placed in position, and any which gave a readable deflexion were rejected.

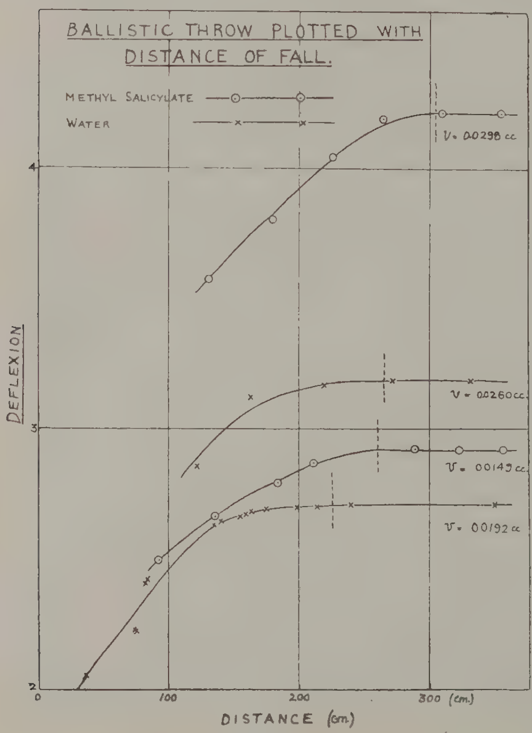


FIG. 2.

The method consisted in allowing drops from a burette to fall on to the filter paper from varying heights and observing the ballistic throw of the balance in each case. It was found that when a series of drops in succession were allowed to form slowly on the end of a tube and detach themselves under gravity there was very little variation in their volume, and any secondary drops formed were so small as to produce no appreciable effect upon the balance.

The individual ballistic throw was taken for a number of drops at each height, and the mean of these values was plotted against the corresponding height (Fig. 2).

A series of drops from the same height were found to give values in agreement to within one or two per cent., the greatest variation being under five per cent. From the curves connecting height of fall and ballistic throw the distance through which a drop of certain volume has to fall in order that it shall attain its constant terminal velocity, may be obtained.

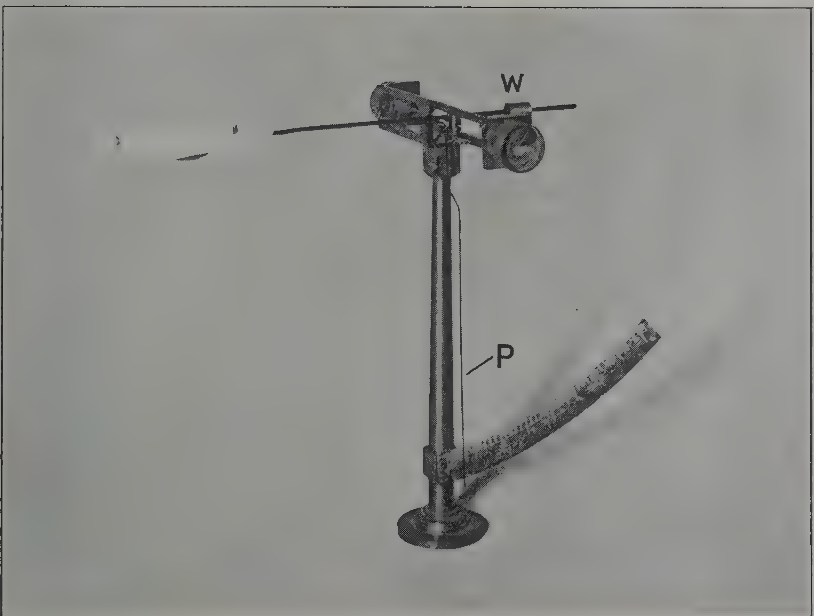


Fig. 1.

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Two liquids were used during the course of these experiments—namely, water and methyl salicylate. The second liquid was chosen because its surface tension and viscosity differed appreciably from those relating to water.

The results obtained for drops varying in volume from 0·2 c.c. to 0·7 c.c. in the

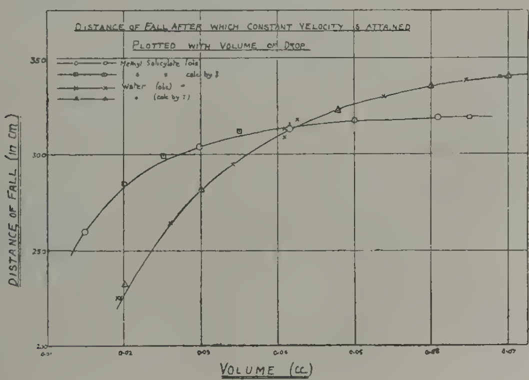



**Fig. 3.**

case of water, and 0·15 c.c. to 0·65 c.c. in the case of methyl salicylate, are given in Table I (Fig. 3).

TABLE I.

| Water. | | Methyl Salicylate. | |
|---|---|---|---|
| Density ... 1·0 gm./c.c.<br>Surface tension 73·5 dynes/cm.<br>Viscosity ... 0·01142 c.g.s. | At 15°C. | Density ... 1·19 gm./c.c.<br>Surface tension 38·8 dynes/cm.<br>Viscosity ... 0·03836 c.g.s. | At 15°C. |
| Volume of drop. | Distance of fall for constant velocity. | Volume of drop. | Distance of fall for constant velocity. |
| C.c. | Cm. | C.c. | Cm. |
| 0·0688 | 340 | | |
| 0·0645 | 338 | 0·0608 | 319 |
| 0·0538 | 330 | | |
| 0·0425 | 318 | 0·0414 | 313 |
| 0·0415 | 316 | | |
| 0·0408 | 313 | 0·0298 | 304 |
| 0·0408 | 309 | | |
| 0·0340 | 295 | 0·0149 | 260 |
| 0·0260 | 264 | | |
| 0·0194 | 225 | | |
| 0·0192 | 225 | | |

DISCUSSION OF RESULTS.

There is no obvious relation connecting the volume of a drop and the distance it has to fall through in order to attain its constant terminal velocity; but from

the results given in Table I an empirical relation has been deduced, which may be written as follows:

$$z=c(1-e^{-Bv}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Thus if

$$\left.\begin{array}{ll} v\to 0 & z\to Bcv \\ v\to\infty & z\to c \end{array}\right\}$$

In this equation $Z$ is the distance a drop of volume $v$ c.c. has to fall through in order that it shall attain its constant terminal velocity, whilst $c$ and $B$ are constants depending upon the liquid employed.

The values of $c$ and of $B$ are 319·2, 0·5527 respectively in the case of water, and 348·2, 1·142 in the case of methyl salicylate. It is interesting to note that the value of the constant $c$ for the second liquid may be obtained by multiplying the value for water by the square root of the density, and the value of $B$ by multiplying the corresponding one for water by the product of the square root of the density and the inverse ratio of the surface tension of the second liquid relative to water at 15°C. These ratios, however, have no theoretical basis and cannot therefore be assumed to apply to the general case.

It will be seen (Fig. 3) that the above equation gives fair agreement for the distance $Z$ over the range of drops investigated, the largest variation being in the case of the smallest drops used.

## The Determination of the Constant Terminal Velocity.

This part of the investigation involved the determination of the constant terminal velocity of drops of each of the two liquids used in the preceding experiments and for a similar range of volume of drop. In the present determination a photographic method has been employed, the final development of which depended essentially on allowing a number of drops, all of the same size, to fall just in front of a highly and evenly illuminated vertical ground glass slit, and photographing them by means of an ultra-rapid cinema-camera. An illuminated time-indicator, placed by the side of the slit, is also included in the photographs.

The resulting photograph (Plate I) thus consists of a succession of images of a particular drop as it passes in front of the slit and of the time indicator. Hence, by measuring the distance between successive positions of the drop, and noting the time interval between each, it is possible to calculate the required terminal velocity. Since the time of exposure is appreciable, the drop appears as a short line on the negative.

## Experimental Details.

The slit used was 30 cm. long and 4 cm. wide, and a number of fine parallel lines, a centimetre apart, were drawn across it to enable the distance that the drop had fallen, between successive photographs, to be easily measured. The time indicator consisted of an electric motor (the speed of which was controlled by an external resistance) carrying a disc, the radius of one half being 7·5 cm. and that of the other half 8·0 cm., rotating in front of another disc which was fixed.

This latter disc had a radius of 9·5 cm., and a semi-circular slit cut in it of radii 7·5 cm. and 8·0 cm.; the periphery of the disc outside the slit being divided into

PLATE I.

*To face page* 170]

ten equal divisions. A piece of tracing paper was fastened behind the slit and division marks to give even illumination.

At any instant a certain portion only of the illuminated semi-circular slit, determined by the position of the rotating disc, was visible. A tachometer driven by a flexible drive from the motor spindle enabled the speed of the rotating disc to be determined with an accuracy of 0·25 per cent. Hence, this speed $W$ being known in r.p.m., the time interval represented by the displacement of one division on the periphery is found to be $3/W$ sec. The distance $H$ having been measured, a drop has fallen between two photographs, and the number of time divisions corresponding to the terminal velocity may be determined. To obtain a succession of drops of the same size, two methods were employed.

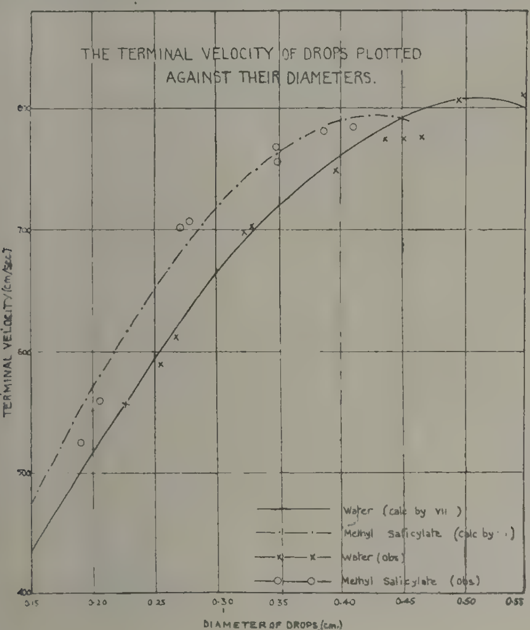



FIG. 4.

1. The attachment of a jet of appropriate size to a Mariotte tube of 500 c.c. capacity.

2. The attachment of a jet of appropriate size to a reservoir of large cross-section 30 sq. cm., so that the variation in head of the liquid during an experiment was very small.

In order to reduce the length of film employed to a minimum, the rate of flow was arranged to be as rapid as possible consistent with the independent formation of the drops.

The jets used were of drawn out glass tube, and by varying the size of the jet drops of different volumes were obtained. The dropping apparatus was mounted at a height of more than 350 cm. above the slit, since all of the drops used, both of water and methyl salicylate, attain a constant terminal velocity after having fallen through this distance (vide (1) Part I).

The drops were allowed to fall through a long pipe in order to eliminate as far as possible promiscuous disturbances of the air. To obtain the best possible contrast on the negative, both liquids were dyed a deep red, Rhodamine Red being the dye used.

The drop size was obtained by measuring the volume occupied by a large number of drops and taking the mean. By observing the drops during their formation through an Ashdown rotoscope, some were seen to have a small satellite drop following them. In some cases two satellites were observed. These satellite drops were caught on an absorbent paper, "Attracta Paper," and their volume estimated from the size of the splash produced. Hence the volume of the drop photographed

was obtained, and from this its diameter calculated, which latter was taken as that of a sphere of volume equal to that of the drop.

The terminal velocities were obtained for drops varying in volume from 0·005 c.c. to 0·07 c.c. in the case of water, and 0·0035 c.c. to 0·04 c.c. in the case of methyl salicylate, and are given in Table II (Fig. 4).

Values of the terminal velocity obtained by Lenard and by Schmidt are given for purposes of comparison.

TABLE II.—*Water.*

| Volume of drop. C.c. | Diameter of drop. Cm. | Terminal velocity. | | |
|---|---|---|---|---|
| | | (Flower) Cm./sec. | (Lenard) Cm./sec. | (Schmidt) Cm./sec. |
| 0·0872 | 0·550 | ... | 800 | ... |
| 0·0830 | 0·548 | 810·7 | ... | ... |
| 0·0669 | 0·510 | 806·4 | ... | ... |
| 0·0655 | 0·500 | ... | 800 | ... |
| 0·0504 | 0·465 | 777·6 | ... | ... |
| 0·0477 | 0·450 | ... | 800 | ... |
| 0·0471 | 0·450 | 774·3 | ... | ... |
| 0·0414 | 0·435 | 774·9 | ... | ... |
| 0·0335 | 0·400 | ... | 770 | ... |
| 0·0287 | 0·385 | 748·4 | ... | ... |
| 0·0225 | 0·350 | ... | 740 | 740 |
| 0·0178 | 0·328 | 702·4 | ... | ... |
| 0·0168 | 0·322 | 697·2 | ... | ... |
| 0·0141 | 0·300 | ... | 690 | 692 |
| 0·0100 | 0·267 | 611·5 | ... | ... |
| 0·0086 | 0·254 | 589·5 | ... | ... |
| 0·0082 | 0·250 | ... | 640 | 638 |
| 0·0060 | 0·226 | 556·7 | ... | ... |
| 0·0042 | 0·200 | ... | 590 | 577 |
| 0·0018 | 0·150 | ... | 570 | 498 |

*Methyl Salicylate.*

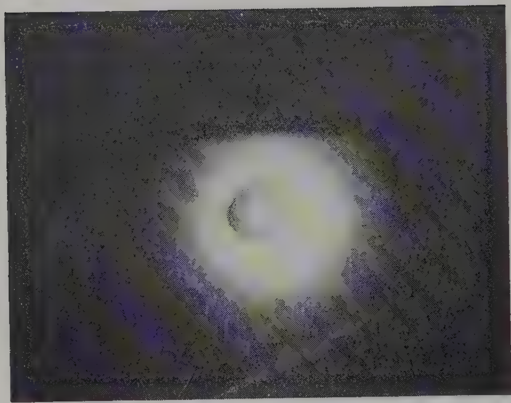
| Value of drop. C.c. | Diameter of drop. Cm. | Terminal velocity. Cm./sec. |
|---|---|---|
| 0·0377 | 0·419 | 784·5 |
| 0·0300 | 0·386 | 781·1 |
| 0·0222 | 0·349 | 754·7 |
| 0·0216 | 0·348 | 767·4 |
| 0·0213 | 0·347 | 767·4 |
| 0·0155 | 0·309 | 729·3 |
| 0·0109 | 0·277 | 706·9 |
| 0·0101 | 0·270 | 701·5 |
| 0·0045 | 0·205 | 559·9 |
| 0·0036 | 0·190 | 522·7 |

## DISCUSSION OF RESULTS.

The probable errors in the above determinations may be dealt with under the following headings :—

(1) Determination of drop size.
(2) Accuracy of measurement of the time interval.
(3) Accuracy of measurement of the distance fallen by a drop.

(1) The volume occupied by a certain number of drops was measured when



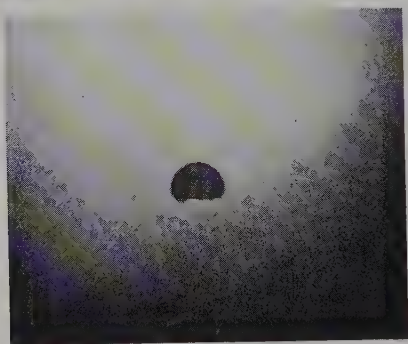


PLATE II.

[*To face page* 173

the drop-rate was about 100 per minute, whereas the drop-rate during an experiment may be as large as 200 per minute in the case of the larger size drops.

Donald[4] and Fildes and Smart[5] have shown that there is an increase in drop-size with drop-rate, hence it is seen that the drop whose velocity was measured was slightly larger than the one for which the diameter had been calculated. Curves connecting the drop-rate and drop-size were therefore plotted for a number of drops, and the error introduced may be safely estimated as not greater than six per cent.

(2) It was possible to measure to 0·1 division on a negative with accuracy, and since this, at the speed used (1500-1800 r.p.m.) represents less than 0·002 second—not more than 2 cm. per sec.—if it is assumed that the terminal velocity of any drop is not greater than 1,000 cm. per second, it may be neglected.

(3) The error introduced here may be computed by calculating the probable error of the arithmetic mean of the whole series of observations for one drop. In this way the greatest percentage probable error of the velocity was found to be less than $\pm 1{\cdot}5$ per cent. The probable mean errors for four velocity determinations taken at random are given below (Table III).

TABLE III.

| Liquid. | Number of observations. | Terminal velocity in cm./sec. | Percentage probable error. |
|---|---|---|---|
| Water ... ... | 14 | 806·3 | ±0·77 |
| | 8 | 774·9 | ±0·35 |
| Methyl salicylate ... | 16 | 729·3 | ±0·658 |
| | 10 | 706·9 | ±1·37 |

From a consideration of the preceding, it is estimated that the present results are accurate to within $\pm 5$ per cent. It will be observed from Fig. 4 that the results for water drops above 3 mm. in diameter differ slightly from those obtained by Lenard, yet, from here to the lower limit of the present determination, there is a large discrepancy in the values. This may possibly be due to errors inherent in Lenard's method, for he was only able to keep his drops suspended for from 2 to 4 seconds before they glided out of the airstream, after which he measured the air velocity at the point of suspension by means of a small spherical cup anemometer. It is quite possible that this latter operation would disturb the air flow and a fictitious value for the velocity be recorded.

## THEORETICAL CONSIDERATIONS.

The constant terminal velocity of fall of very small drops—0·1 mm. or less in diameter—may be computed by Stokes's well-known equation;[6] but for drops with larger diameters this relation ceases to hold. These larger drops become distorted during their fall, and expose a larger surface to the air than would the corresponding spheres of equal volume. The accompanying photographs (Plate II) show the distortion brought about in a water drop 0·464 cm. in diameter after it had acquired its terminal velocity. Neither the question of the distortion of the drop, nor the effect of this upon its terminal velocity, appears to lend itself to simple mathematical treatment, and the equations of fall for these larger sized drops are but little more than empirical.[7] It is possible, however, to write down an equation

of motion for a drop falling freely through air, provided that certain assumptions are made. If it is assumed that the resistance to the motion of the drop due to the air is proportional to the square of the velocity, and that the effect of the air which is dragged along with the drop is negligible, the equation of motion for constant terminal velocity will be

$$g(\rho-\sigma)v-kV^2=0 \qquad (2)$$

where $V$ is the constant terminal velocity,
$\rho$ the density of the liquid,
$\sigma$ the density of air,
$v$ the volume of the drop,
$k$ the coefficient of resistance depending upon the size of the drop.

Now, since $\sigma$ is small compared with $\rho$, it may be neglected in the above equation.

Thus $$V^2=g\rho v/k \qquad (3)$$

If now the value of $k$ be determined by substituting the observed value of the terminal velocity in (3), it is found to increase with the size of the drop (Table IV). The value of $k$ for solid spheres of the same volume, deduced from values of the terminal velocity of steel balls obtained by Richardson,[8] are given for purposes of comparison.

TABLE IV.—*Water.*

| Volume of drop. C.c. | Diameter of drop. Cm. | Terminal velocity. Cm./sec. | $k$ Gm./cm. | $k$ Gm./cm. (Steel Spheres). |
|---|---|---|---|---|
| 0·0830 | 0·548 | 810·7 | $12{\cdot}40\times10^{-5}$ | $8{\cdot}0\times10^{-5}$ |
| 0·0669 | 0·510 | 806·4 | 10·07 | 7·0 |
| 0·0504 | 0·465 | 777·6 | 8·17 | 5·8 |
| 0·0471 | 0·450 | 774·3 | 7·70 | 5·5 |
| 0·0414 | 0·435 | 774·9 | 6·76 | 5·2 |
| 0·0287 | 0·385 | 748·4 | 5·02 | 4·2 |
| 0·0178 | 0·328 | 702·4 | 3·53 | 3·5 |
| 0·0168 | 0·322 | 697·2 | 3·38 | 3·2 |
| 0·0100 | 0·267 | 611·5 | 2·62 | 2·7 |
| 0·0086 | 0·254 | 589·5 | 2·32 | 2·5 |
| 0·0060 | 0·226 | 556·7 | 1·92 | 2·0 |

*Methyl Salicylate.*

| Volume of drop. C.c. | Diameter of drop. Cm. | Terminal velocity. Cm./sec. | $k$ Gm./cm. | $k$ Gm./cm. (Steel Spheres). |
|---|---|---|---|---|
| 0·0377 | 0·419 | 784·5 | $7{\cdot}15\times10^{-5}$ | $5{\cdot}0\times10^{-5}$ |
| 0·0300 | 0·386 | 781·1 | 5·74 | 4·2 |
| 0·0222 | 0·349 | 754·7 | 4·54 | 3·8 |
| 0·0216 | 0·348 | 767·4 | 4·18 | 3·8 |
| 0·0213 | 0·347 | 767·4 | 4·23 | 3·8 |
| 0·0155 | 0·309 | 729·3 | 3·40 | 3·0 |
| 0·0109 | 0·277 | 706·9 | 2·55 | 2·8 |
| 0·0101 | 0·270 | 701·5 | 2·39 | 2·7 |
| 0·0045 | 0·205 | 559·9 | 1·67 | 1·9 |
| 0·0036 | 0·190 | 522·7 | 1·54 | 1·8 |

A relation between $k$ and $r$, the radius of the drop, has been obtained which is given below:

$$k^2(P-r)=Qr^3 \qquad \ldots \ldots \ldots \ldots \quad (4)$$

where $P$ and $Q$ are constants depending upon the liquid employed. The values of $P$ and $Q$ are respectively 0·339, $555\times10^{-10}$ in the case of water, and 0·285, $415\times10^{-10}$ in the case of methyl salicylate. If the value of $k$, in terms of radius $r$, from (4), be substituted in (3), the terminal velocity will be given by

$$V^2=(4/3)\pi\rho g[r^3(P-r)Q^{-1}]^{\frac{1}{2}} \qquad \ldots \ldots \ldots \ldots \quad (5)$$

By differentiating this equation it is found that $V$ has a maximum value when $r=0{\cdot}75P$ cm. From this it will be seen that water drops for which $r=0{\cdot}25$ cm., and methyl salicylate drops for which $r=0{\cdot}21$ cm., have the most rapid fall. Lenard also found that there was a maximum value for the terminal velocity when the radius of the drop was 0·25 cm. in the case of water.

The explanation of the above, given by Lenard, is that the friction of the air causes deformation of the drops, so that instead of retaining the shape of spheres they became flattened out, thus presenting an increased surface to the air through which they are falling. This deformation is slight for the smaller drops, but increases rapidly in the case of the larger drops. After a certain size of drop has been attained, any further increases in volume produces a greater flattening, and instead of the velocity being increased it is slightly decreased.

## Acknowledgments.

My thanks are due to Mr. J. D. Fry for his suggestions and for the interest he took during the progress of this work, and to Mr. F. J. W. Whipple I am indebted for his assistance in the final preparation of this Paper.

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## DISCUSSION.

Mr. J. H. Coste : The relatively low terminal velocities found by Mr. Flower are interesting in connection with the dissolved oxygen content of rainwater which E. Hannaford Richards has shown to be frequently less than corresponds to saturation at the temperature of collection.

It has to be remembered that raindrops are formed in a relatively attenuated atmosphere, where the partial pressure of oxygen is low, and hence the amount needed for saturation of water is less than at the ground level. Nevertheless, it is somewhat remarkable that a drop should not become saturated with atmospheric gases during its fall. The process used by Richards for determination of dissolved oxygen was trustworthy, and the errors of collection would tend to bring the water into closer equilibrium with the air.

The following are some of Richards' results for the oxygen dissolved in rainwater collected at Rothamsted in 1915 : January (80 per cent. saturation), 79·2 The remaining results are for 90 per cent. saturation, as follows : January, 87·0 ; February 85·7 ; July, 81·0 ; 84·5, 89·7,

August, 88·2; November, 89·5. There were 26 results in all, and only 6 over 98 per cent. of saturation. It should be noted that 1,000 ml. of distilled water dissolves about 10·2 ml. of oxygen from moist air at 0°C. and 6·4 ml. at 20°C. The table is from Jour. Agric. Sci., VIII, iii, 1917, p. 334, and the figures have been corrected by me for a pressure of dry air of 760.

Dr. W. N. Bond: Assuming that the circulation of the liquid inside the drop has little effect on the terminal velocity, certain results may be predicted. We obtain by dimensional considerations

$$g(\rho-\sigma)v = V^2r^2\sigma \, . \, f\left(\frac{Vr\sigma}{\mu} \quad \frac{r^2(\rho-\sigma)g}{T}\right)$$

(where $T$ is the surface tension).

Firstly, for small drops the product involving T may be omitted. Thus, for small drops of two liquids falling in air, if $Vr$ be the same, we might expect $V^2/r(\rho-\sigma)$ also to be the same.

Secondly, the radius near which the transition due to flattening of the drop occurs should be of the order given by $r^2(\rho-\sigma)g = T$. A critical radius of this order has been found for air bubbles rising in water, there being a maximum terminal velocity for such sizes of bubble. Also in the case of the terminal velocity of drops and bubbles moving in viscous liquids (the kinetic energy being negligible) I have found in some conjoint work a critical radius of this order, but not involving a flattening of the drops. (Phil. Mag., April, 1928.)

In Mr. Flower's experiments, since $Vr\sigma/\mu$ is not greatly different for the drops of maximum terminal velocity in the two cases, I should have expected the critical sizes to have been related by the proportionality $r$ (critical) $\propto \sqrt{T/(\rho-\sigma)}$.

Mr. R. G. Lunnon (communicated): Mr. Flower's methods are ingenious, and his results are of real value. They show the resistance for small drops to be much greater than had been supposed, for the values of the resistance coefficient ($W/\rho v^2d^2$) are between 0·15 and 0·20 for this range of Reynolds' numbers for perfect spheres, and the present values, for deformed spheres, are from 50 per cent. to 100 per cent. greater. From the photographs, it does not appear that the flattening was very great, and I think that an increase of 20 per cent. in the transverse diameter might account for the extra resistance. Were there no indications of the drops oscillating about a mean spherical shape? It may be noted that the $v^2$ resistance law is not applicable even for perfect spheres, with these speeds; the power of $v$ should be between 1·4 and 1·7. It would be useful to know how the satellite drops were separately measured, and how large they were.

Mr. W. A. Benton suggested that the velocity might be found by measuring the ballistic deflection, though the relation between these quantities would not be linear.

Author's reply: Replying to Dr. Bond, I would like to point out that from the experimental values obtained [$r$ (critical, water)/$r$ (critical, methyl salicylate)] is 1·2, and if we substitute in his expected proportionality we get the value 1·5, which is more or less in agreement. Replying to Mr. Lunnon, the drops were observed only for a distance of about 30 cm., and there were indications of the drops oscillating about a mean spherical shape. It was thought that as the drops became distorted the oscillations would decrease and tend to be about a mean flattened-spherical shape. This might explain why the distortion in the cases shown is not quite the same in each photograph. The satellite drops were blown by a gentle horizontal stream of air away from the line of fall of the larger drops on to the absorbent paper, and thus they could be measured separately. The largest satellite drop measured had a diameter $<0{\cdot}1$ cm. Replying to Mr. Benton, I might say that the original intention was to obtain the terminal velocities from the ballistic deflection, but since the calibration curve was very steep the method, with the form of apparatus used, was not practicable and for that reason the photographic method was employed.

# XXII.—ON THE LONGITUDINAL WAVE ALONG A ROD.

By SATYENDRA RAY, *M.Sc.*, Lucknow University.

*Received February* 8, 1928.

(*Communicated by* PROF. LALJI SRIVASTAVA, Government College, Ajmer, India.)

ABSTRACT.

That longitudinal waves are propagated with the same velocity for all wavelengths only when the waves are " geometrically similar " is here proved in exactly the same manner as for either strings, air columns or transverse waves along stretched strings. The expression for the velocity is of course different.

HOUSTOUN points out in his Introduction to Mathematical Physics, §98, that the longitudinal vibrations in a rod of any section are mathematically the same as those for the transverse vibration in a stretched string. An error due to contraction of the cross-section during elongation, and vice versa, is, however, pointed out as leading to an error that " can be neglected."

We may proceed to correct for this error for a rod of any section of area $a$. The elongation per unit length is $\partial\xi/\partial x$ at any section $A$, and the resultant force on a slab bounded by two planes $A$ and $B$, distant $dx$ apart, is equal to

$$Ea\left(\frac{\partial\xi}{\partial x}+\frac{\partial^2\xi}{\partial x^2}\,.\,dx\right)-Ea\frac{\partial\xi}{\partial x}=Ea\frac{\partial^2\xi}{\partial x^2}\,.\,dx$$

where $E$ is Young's modulus. We equate this to the expression $m\,.\,\partial^2\xi/\partial t^2$, where $m$ is the mass of the slab. In obtaining an expression for $m$ we remember that the cross-section $a$, owing to a contraction equal to $\sigma$ times the elongation in the linear dimension, becomes

$$a(1-\sigma\partial\xi/\partial x)^2$$

We therefore have for the equation of motion

$$\rho a\left(1-\sigma\frac{\partial\xi}{\partial x}\right)^2\,.\,dx\,.\,\frac{\partial^2\xi}{\partial t^2}=Ea\frac{\partial^2\xi}{\partial x^2}dx$$

or

$$\frac{\partial^2\xi}{\partial t^2}=E\,.\,\frac{\partial^2\xi}{\partial x^2}\Big/\rho\left(1-\sigma\frac{\partial\xi}{\partial x}\right)^2$$

This expression leads approximately to the form

$$v=v_0(1-\sigma\partial\xi/\partial x)^{-1} \qquad \text{. . . . . . . . . . . . (1)}$$

for the velocity $v$ of propagation, where $v_0{}^2=E/\rho$

In case $\xi$ is of the form $\xi=a\sin 2\pi x/\lambda$

$$\frac{\partial\xi}{\partial x}=\frac{a}{\lambda}\cos 2\pi\frac{x}{\lambda}$$

so that the value of $\delta\xi/\delta x$ in (1) is a function of $a/\lambda$.

We note that, as in the case of waves along electric tubes of force, air columns and stretched strings vibrating transversely, $v$ is the same for all values of wave-length if the ratio $a/\lambda$ between the amplitude and wave length be constant—i.e., if the waves be geometrically similar.

# XXIII.—ON THE DAMPING OF MERCURY RIPPLES.

By J. J. Manley, *M.A.*, Research Fellow, Magdalen College, Oxford.

Abstract.

This brief communication is descriptive of a method devised for suppressing wavelets set up by temporary vibrations upon mercurial surfaces.

In a paper* recently read to the Society I dealt with an interferometer pressure gauge in which the reflectors were the summits of two mercurial columns. Attention was drawn to the necessity for supporting the instrument upon a bench of great stability; mention was also made of the ease with which minute ripples are formed upon the mercury by passing vehicles, whether near or somewhat distant.

As a result of numerous trials, it is now possible to describe a plan whereby the difficulties arising from transitory vibrations may be overcome.

Experiment and theory agree in showing that when a glass plate of ordinary thickness is placed upon mercury the glass, like other solids, quickly makes contact with the wall of the containing vessel. But when the plate does not exceed 2 mm.† in thickness its behaviour is completely reversed. Such a plate instantly moves away from the wall, and, provided the width of the cistern holding the mercury is but slightly in excess of that of the disc, the latter is at once automatically and correctly centred upon the pool.

This extremely convenient and helpful behaviour of a thin plate is not apparently generally known, and all my efforts to discover some allusion to it have hitherto failed.

In view of the fact just mentioned, boxwood cisterns, shallow and circular, and having diameters ranging from 27 to 35 mm., were prepared. These were partially filled with pure mercury, and during experiments with any one of them the mercury was almost covered with a parallel-plane glass disc 25 mm. in diameter. Trials were made in the following way.

First, the cistern to be tested was placed upon the interferometer bench and a powerful beam of light directed upon the glazed mercurial surface, whence it was reflected to a distant screen. Next, the cistern was somewhat violently disturbed, and the time required by the mercury to come to rest noted. Proceeding thus with the several cisterns, it was quickly found that the most rapid and effective damping resulted from the use of one whose diameter closely approximated to that of the disc—a result in accord with theory.

Use was now made of another cistern having an internal diameter of 26 mm.

* Proc. Phys. Soc., Vol. 40, Pt. 2, p. 57 (1928).

† The thickness of the plates used for the experiments here described was 1·99 mm. Plates having a thickness of 2·2-2·4 mm., according to the kind of glass, exhibit when placed upon mercury complete neutrality: they are neither attracted nor repelled by the walls of the containing vessel. Hence the necessity for restricting the thickness to 2 mm. or less. The thinner the plate the more prompt its movement to the centre of the pool.

Thus the annular clearance between cistern and disc was reduced to 0·5 mm. With this apparatus the effects of the most severe shocks were suppressed within two seconds or less. The efficacy of this particular cistern was still further tested by viewing the reflected image of a pair of cross-threads through a reading telescope; and again the damping, as thus determined, was equally rapid and complete. But, in order to ascertain to what degree of perfection the damping had been raised, it was obviously necessary to try the device by means of the interferometer itself.

Accordingly, yet another cistern similar in all respects to that just dealt with was made, partially filled with mercury, and, like the other, equipped with a parallel-plane glass disc. These twin cisterns were then used as the two reflectors of the interferometer.

The result of a trial was disappointing, for from the movements of the interference fringes it was seen that transitory vibrations caused the glazed mercury to oscillate in a complicated manner; and, further, that the oscillations, though small, were slow in dying away. Hence the next task was an attempt to introduce some additional device whereby this final difficulty might be overcome. Success was attained in the following way.

First, the glass discs were cleaned with alcohol, and then to some 6 or 8 equally spaced points upon the circumference of each plate was applied a droplet of an alcoholic solution of shellac. Next, a thread known as "silk twist," having a thickness of 0·5 mm., was immediately carried round the disc and through the several portions of shellac. As the solution hardened the thread was firmly secured. Finally, the internal diameter of the cisterns was suitably enlarged for the reception of the encircled plates. The annular clearance between plate and cistern was now less than 0·5 mm. The use of silk twist was here introduced for the following reasons.

When the thread is gently rubbed its smoothness is either increased or lessened, according to the direction of the stroke. A decrease in smoothness is due to numerous projecting filaments of great fineness and delicacy. In order that advantage might be taken of this property, the encircling twist was appropriately rubbed, and the discs thus armed with outstanding filaments or tentacles; and these, when the discs were *in situ*, lightly but surely clung to the inner face of the opposing wall. It will therefore be seen that, although the floating discs were thus effectively moored, they were yet free to respond to and follow any changes in the level of the mercury.

On subjecting the now completed device to a series of tests with the interferometer it was seen that the passing of ground tremors produced nothing more than simple and momentary swayings of the floating discs. Generally, the movements were such that the interference fringes oscillated evenly. Initially represented by one or two fringes only, the oscillations quickly died away.

From the results of the final exacting tests the conclusion reached was that the plan evolved for the suppression of mercurial ripples was effective, and sufficient to meet the demands of the experimenter.

We conclude by observing that an excessive control is indicated by a failure of the plates to respond to temporary vibrations. Inadequate control is, on the other hand, shown by a lack of the necessary damping. In the one case the silk tentacles are too many and in the other too few. The first named defect may be removed by a fine platinum wire used as a smoothing tool, and for convenience

applied in turn to several evenly distributed regions of the attached silk twist. The other defect is similarly remedied with the aid of a very thin wooden rod which, unlike the wire, acts when passed over the thread in the right direction, as a roughing tool, and so increases the number of projecting filaments or tentacles. The art of regulating the number of active tentacles may be acquired with ease.

Finally, it may be remarked that the method of damping described above, might with considerable advantage be used in other instances ; as, for example, in the case of artificial horizons.

# XXIV.—EXPERIMENTS WITH MERCURY JETS AND THE PHENOMENA EXHIBITED AT THEIR IMPACT WITH STEEL AND GLASS.

By Professor WILL C. BAKER, *M.A.*, Physics Department, Queen's University, Kingston, Ont., Canada.

*Received March* 9, 1928.

ABSTRACT.

As a light sphere is retained in a vertical jet of fluid in virtue of the change of momentum of the fluid produced by its adhesion to the sphere, it was thought that a steel sphere would not be retained in a vertical mercury jet, as there is no " wetting " of the steel by that fluid. Experiment showed that a given bicycle ball might or might not be retained by such a jet, as the speed of the jet (at a given angle of incidence) rose above or fell below a critical value for that ball.

Conditions were simplified by the use of cylindrical and of plane surfaces of steel, and an approximately constant time of adhesion between mercury and steel was found for various speeds of impact. This led to the explanation of the phenomenon in terms of the well-known instability of jets.

IN a note " On the Retention of a Ball by a Vertical Water-jet,"* the author showed that the phenomenon was essentially due to the change of momentum imparted to the water by its adhesion to the already wet ball.† It appeared that a consequence of this would be that, since mercury " wets " neither glass nor steel, balls of these materials would not be retained in a vertical mercury jet. A trial showed, however, that under certain conditions such balls could be retained, and experiments were begun in order to investigate the phenomena.

The jet was produced and controlled in the following way :—

From a glass separating funnel—which was used as a mercury reservoir—a quill tube ran downwards for about 75 cm. It was then bent upwards to a glass stop-cock lubricated with graphite. Just above the stop-cock the tube turned horizontally to a nozzle that allowed of adjustment to any angle in a vertical plane. The nozzle was sometimes of steel and sometimes of glass. To determine the velocity of the jet as it left the nozzle under any given adjustment of the stop-cock, the stream was caught in a small beaker for a known number of seconds, and the required velocity was computed from the volume issuing per second and the measured area of the nozzle. . A better method of estimating the effective area of the nozzle was developed later. (See below.)

Instead of measuring the force exerted by the jet on the ball, as had been done in the case of the water jet,‡ it was found convenient to use the jet itself as an indicator of the forces acting. The vertical jet was allowed to impinge at different speeds and at different angles of incidence on a fixed ball. At low speeds the jet flattened out in a small nearly elliptical sheet after it struck the sphere ; it then gathered itself together and left the surface at an angle as shown in Fig. 1a. This

* Science, Vol. LXIV, No. 50, p. 161.

† The same explanation was put forward by Osborne Reynolds (*see* Collected Papers, Vol. I, p. 1), but no experimental demonstration of its correctness was there given.

‡ Loc. cit.

behaviour of the jet implied the action on the ball of a force directed away from the axis of the jet. At high speeds the elliptical sheet of mercury formed a sort of cap on the surface of the ball, and the jet left from the upper end of the cap as indicated in Fig. 1b. Obviously the reaction in this case urged the ball *towards* the axis of the jet. Intermediate speeds showed cases where the fluid apparently left the surface in the line of impact as shown in Fig. 1c.

To get simpler conditions, a horizontal jet was used with a polished cylinder of steel (radius 0·7 cm., axis vertical). At the highest available speeds the fluid would adhere to the steel through an arc of 150° before gathering itself into a spray as it left the surface.

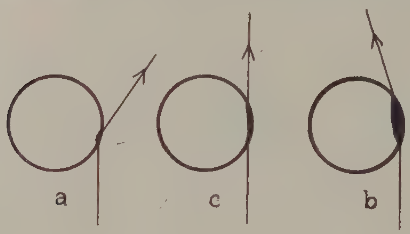

FIG. 1.

The mercury against the cylinder was a thin flat elliptical sheet bounded by a thicker roll at the edges. The results were not sensibly different when the jet and cylinder were placed in a jar and the air exhausted with a motor-driven rotary oil air pump, and this fact was taken to show that the phenomenon did not depend on the jet's setting up reduced pressures by any entrainment of the air in contact with its surface. The higher the speed of the jet the longer was the major axis of the "ellipse of contact," and as the relation between these quantities was apparently linear, it was thought that perhaps the mercury adhered to the steel for a definite time—about 1/200 of a second—before breaking away. (See below.) There were strong electrical effects due to the breaking of the jet into spray as it left the cylinder, but while the spray had a powerful effect on an electroscope when the air was present, the same electroscope showed no measurable potential difference between the steel nozzle and the jet, nor between the jet and the steel cylinder. It was thought that any electrical effect sufficient to cause the observed change of momentum in the mercury would surely have been great enough to affect the electroscope. A glass cylinder freshly cooled in air from a bright red heat showed the same phenomena as did the steel, so the idea of the adhesion being due to films of air on the glass was set aside.

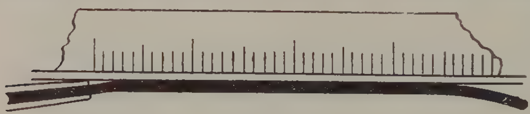
FIG. 2.

To simplify conditions further the jet was caused to impinge on the under side of a flat horizontal steel blade (Fig. 2). For high speeds the mercury spread out into a flat elliptical sheet as before, and at the further end gathered into a jet which, of course, resolved itself into drops almost at once. A paper scale attached to the upper side of the blade made it possible to read the distance over which the mercury ran in contact with the steel. At the highest speeds available (about 375 cm./sec.) the major axis of the ellipse was about 2·3 cm. long, but as the speed was reduced the dimensions of the elliptical sheet diminished until at 125 cm./sec. its length was only 1 mm. A further cutting down of speed at this point produced a sharply defined change in the behaviour of the jet. The flattened portion disappeared,

and the mercury ran along the under side of the steel for 3 or 4 cm. before leaving in a stream of drops. The length of this type of run on the steel also diminished with slower jet-speed, until at about 40 cm./sec. its contact with the steel was of the order of 1 mm. The column of mercury running on the under side of the steel was usually a nearly uniform cylinder right up to the point where it broke away. Under some conditions, however, waves could be seen running along it which at certain speeds settled into systems of a stationary type, three or four complete waves being sometimes obtained. These were of a rapidly diminishing amplitude, but were of an apparently constant wave-length. When the impact was on glass it could easily be seen that a vibrating jet was adhering to the solid ; for points at which the depth of the jet was least were points at which its breadth was greatest. The fact that three or four complete waves were formed shows that there was a real adhesion between the mercury and the glass. Again, if a copper wire (say 1 mm. diameter by 8 or 10 cm. long) be well amalgamated and laid on a piece of clean plate glass, with a small drop of mercury to fill up the space between the wire and the glass, and if this be gently tipped to let the excess of mercury run off along the wire, the whole system may be inverted without the wire dropping from the glass. The wire and adhering mercury are held up against their weight by the adhesion of the fluid metal to the glass. The function of the copper wire is to lend its rigidity to the system so that the mercury cannot draw itself up into a drop and cut off its contact with the glass by the " sphincter " action of the tension about the circle where the fluid meets the glass.

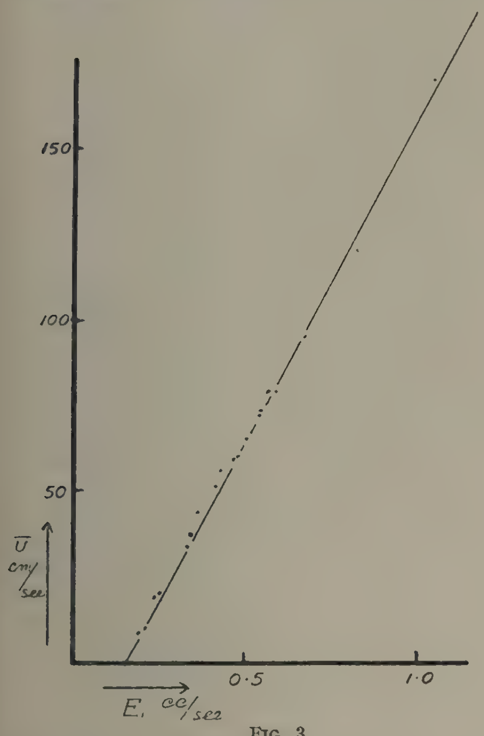


FIG. 3.

This fact, as well as those reported in a note " On the Adhesion of Mercury to Glass Surfaces,"* show that the adhesion is not a temporary phenomenon, as suggested above, and that the effect of adhesion for definite small times in these experiments has still to be sought.

The jet velocities as noted above were determined from a nozzle area computed from the average of a series of measured diameters of the orifice. To check these, and to find whether any contraction lowered the effective area of the cross-section, the jet was allowed to discharge horizontally past a sheet of squared paper,

* Science, Vol. LXVII, No. 1725, p. 74.

and the initial horizontal speed of the jet was deduced from the co-ordinates of the observed parabolas described by the drops. These velocities, for the range of speeds available, were then plotted against the volume issuing per second. A perfectly linear graph was obtained (Fig. 3), but it did not pass through the origin, so that the newly formed drop has no horizontal velocity for an efflux of 0·15 c.c./sec. This loss of speed the graph shows to be the same at all speeds. It is accounted for by the fact that as each drop forms at the end of the issuing jet it is subjected to an impulse $2\pi T \int r dt$ of sensibly constant integral value, due to the surface tension ($T$) around the neck of the forming drop. The observed constancy of this impulse indicates a fairly constant functional relation between $r$ and $t$ (time) in the integral just given at all speeds (for a given nozzle).

Further, from the graph we see that the horizontal velocity of the drops ($\bar{v}$) is given by an equation of the type

$$\bar{v}=(E-c)K/a,$$

where $K$ and $c$ are constants, $a$ the effective cross-section of the nozzle, and $E$ the efflux in c.c. per sec. Also $a=dE/dv$, which gives a method for the determination of the effective area of the jet. Then $E/a$ gives the velocity with which the mercury issues from the nozzle.

Next, the jet was set so as to deliver the cylindrical column of mercury to the under side of the steel blade as nearly horizontally as possible (as in Fig. 2), and the squared paper was employed, as above, to determine the horizontal velocity of the drops as they broke from the blade after runs of various lengths—i.e., from different nozzle velocities. The graph for the velocities of the drops ($v$) plotted against the efflux ($E$) was of the same nature as that shown in Fig. 3. The effective area of the nozzle and the nozzle velocity ($v$) were therefore determined as above, The nozzle velocity when plotted against the efflux gave a line parallel to the plot of $\bar{v}$· but passing through the origin. This shows that the difference between $v$ and $\bar{v}$ was constant at all speeds. When $v$ and $v$ are plotted against the distance run on the steel, the same type of graph was obtained.

Thus the measurements show no appreciable loss of speed as the mercury runs along the blade,* but they reveal a change in velocity, constant at all speeds, due to the formation of the drops at the end of the cylinder, as in the case of the free jet.

We are thus driven to the view that the mercury adheres to the steel as long as it remains in a cylindrical form. As soon, however, as the mercury gathers itself into spheres, owing to the instability of the cylinder, the action of the surface tension " closes the drop off " by the " sphincter " action about the circle of contact between the mercury and the steel. That is, the force of adhesion between the molecules of the fluid and those of the solid is usually† less than the force due to $T/r$

* This agrees on the one hand with the apparent constancy of the diameter of the cylinder as it passes along the blade, and on the other hand with the apparent constancy of the wave-length of the standing waves when they are present.

† *Usually*, for there is the strange phenomenon of " sticking " reported in a note " On the Adhesion of Mercury to Glass " (Science, Vol. LXVII, p. 74, January 20, 1928). In the experiments there described the air-mercury edge would at times adhere to the glass so as to take an abnormally large pull to cause a change in the line of contact. For instance, in making a series of observations on the shrinking of the circle of contact from some given circle to a new stable position (due to a given displacement of the lens (*see* Fig. 5) of the article quoted), the edge often swept past points at which in a previous trial it had stuck, and at which it could not be got to stick again.

when the circle of contact is small. These views are strengthened by the fact of the adhesion to glass of the amalgamated wire mentioned above, and of the adhesion of the mercury to the surfaces of glass lenses. When the fluid is "closed off," as described, the glass is left "dry," since the cohesion of the mercury is greater than its adhesion to the glass. These considerations show why the run of the mercury on the steel is proportional to the speed, for it depends on the time required for the cylinder to break into drops; the case of the apparently constant time of adhesion of the flat elliptical films is probably fundamentally the same even if the action be much more complex. They give, too, the reason for the results indicated in Fig. 1a, b and c, for if the time of contact be approximately constant the jet for slow speeds will be thrown off before it reaches the "equator" of the ball, while for greater speeds it will be well past that line before being torn away by the formation of drops. It will consequently leave the sphere in this case in an entirely different direction, as indicated in Fig. 1. The differences in the reaction of the jet on the ball are directly dependent on these changes in direction.

## DISCUSSION.

Dr. W. H. Eccles offered an alternative explanation of the ball-and-jet phenomenon. If the jet strikes the ball at a point away from the vertical axis the stream distributes itself unequally over the surface of the ball, and the thinner region of the stream loses velocity faster than the thicker region. Hence the pressure exerted by the former is the greater in accordance with Bernoulli's theorem, and tends to restore the ball to the symmetrical position. The Author had experimented with steel and mercury in order to obtain an unwettable surface, but could he not more easily have waxed the surface of one of the balls ordinarily used? The Paper reminded the speaker of a small slow-motion, variable-speed motor which he had once constructed, in which a water-jet of adjustable volume fell vertically on to a ball fixed to the lower end of an arm, the upper end of which was fixed to a shaft. The ball oscillated in the jet, and the consequent reciprocating motion of the shaft was rectified by means of ratchet mechanism.

Mr. T. Smith said that Dr. Eccles' explanation should be tested quantitatively, since the low-pressure region of the stream rises higher, and so covers a wider area than the low-velocity region. The correctness of his explanation cannot, therefore, be decided qualitatively.

Dr. Ferguson said that he did not think that Osborne Reynolds, who first discussed the problem, had considered Bernoulli's theorem as a possible explanation. It was interesting to note, however, that Reynolds had described an experiment in which a disc forming the bob of a pendulum had been made to vibrate in a water-jet in the way mentioned by Dr. Eccles.

## XXV.—THE ELASTIC CONSTANTS OF GLASS.

By Edgar Philip Perman and William Donald Urry, University College, Cardiff.

*Received March* 18, 1928.

Abstract.

(1) The coefficients of compressibility of soda-glass and Jena $16^{III}$ glass have been determined at six temperatures ranging from 30°C. to 80°C. (2) From experiments on the effect of external pressure only, Poisson's ratio has been determined, and hence the modulus of rigidity and Young's modulus.

In measuring the coefficient of compressibility of a liquid by the application of equal internal and external pressure to a bulb containing the liquid, it was necessary to know the bulk modulus of the glass employed. We have, therefore, determined the coefficients of compressibility of the glass used in constructing the piezometer bulbs, and we have carried out a second series of experiments, from which values have been obtained for the bulk modulus, rigidity modulus and Young's modulus.

The two methods of experiment consist of :—

(1) The measurement of the direct linear compression of a long glass cylinder.

(2) The application of external pressure only to a piezometer bulb, and the measurement of the compression produced.

Two specimens of glass have been used :—

(*a*) Jena (red line) $16^{III}$ standard glass.

(*b*) A typical soda glass such as is commonly used in experimental work.

### Summary of Previous Work.

In the work of Regnault* on the compressibility of liquids pressure was applied in turn to :—

(1) The outside only of the piezometer bulb.

(2) The inside and outside of the bulb simultaneously.

(3) The inside only of the bulb.

Three equations were obtained, but the final solution involved the unjustifiable assumption that Poisson's ratio for glass$=\frac{1}{4}$. The most direct method is that suggested by Amagat,† and used later by Buchanan and Tait.‡ It was used by Amagat up to a pressure of 2,000 atmospheres.

It consists of the direct measurement of the longitudinal contraction of a glass rod under hydraulic pressure. This was observed directly in the experiments of Buchanan and Tait, but by an electrical contact method in the experiments of Amagat.

Another method is to apply a linear tension to a bulb of the glass closed by a graduated capillary. The increase in the internal volume is measured by the

* Memoirs de l'Institut de France, 21, 429.

† Comtes Rendus, p. 727 (1889).

‡ Roy. Soc. Edin. (1880).

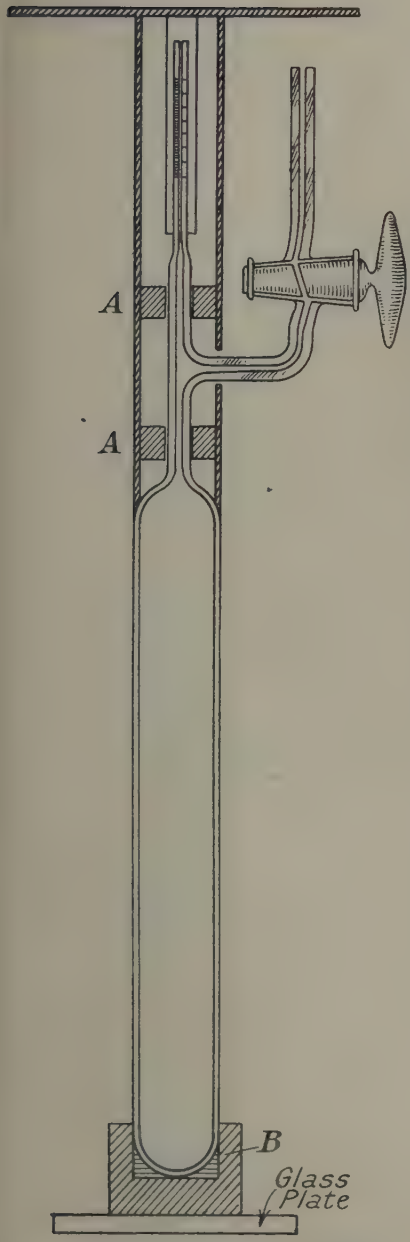


DIAGRAM 1.—GLASS TUBE ARRANGED TO MEASURE COMPRESSIBILITY OF GLASS.

descent of liquid in the capillary. In the last two methods the value obtained is of course one-third of the bulk modulus.

METHOD I.

We have employed a modification of this last method. In applying a linear tension it is necessary to strain some portion of the bulb where irregularities occur, such as a seal between the bulb and the capillary, or an irregularity necessary for clamping. This complicates the evaluation of the factor $V \,.\, dP$ at the various cross-sections. Consequently we have attempted a modification of the above method. A long cylindrical bulb, whose uniform length is great compared with the hemispherical end, is subjected to compression, the force being applied below the capillary seal, which is under no strain. A brass tube of external diameter equal to that of the bulb, and about nine inches in length, is filed to fit the hemispherical end closing the bulb. To the centre of this end of the bulb is sealed a capillary, which in turn is sealed to the indicator capillary and the side tube water level adjuster. To the other end of the brass tube is soldered a table to carry the mass. The brass tube itself is suitably slotted for the side capillary to pass through and to obtain a view of the indicator capillary within. (Diagram 1.)

It will be seen that the pressure is applied at a point below which no serious irregularity occurs. In fact, the conditions approach the ideal conditions of a flat-ended cylinder, the only divergency being one hemispherical end. The factor $V \,.\, dP$ then is made up of :—

(1) $V_1 \,.\, dP_1$ for a flat-ended cylinder; and

(2) $\int_{r=0}^{r=R} V \,.\, dP$, where $R$=the radius of the hemispherical end.

Owing to the great length of the cylinder compared with the radius of the hemispherical end, the application of this integral gives a

result differing negligibly from that obtained in using the total internal volume and the mean value of $dP$. The latter method has therefore been used.

## Experimental Procedure.

The compressibility coefficient for the glass is given by

$$c=3(dV/V)\div\{Mg/\pi(R^2-r^2)\}$$

where $dV$ is the volume decrease,
$V$ the total initial volume,
$R$ the external radius,
$r$ the internal radius,
$Mg$ the force applied.

Prior to a set of experiments, $V$ was first determined by filling with distilled water up to the point of seating of the brass tube and adjusting at 30·00°C. From weighings and density tables $V$ was obtained. The entire apparatus was then filled with water, and the brass tube, fitted with rubber guiders $AA$ lubricated with liquid paraffin, placed in position. The base of the cylinder was fitted into a metal base $B$. The apparatus was immersed in an electrically controlled thermostat constant to 1/300°C.; with this sensitivity the water level in the indicator capillary remained constant for a far greater time than that required to perform a complete reading.

A mass of 5,000 grammes was placed on the platform at the top of the tube in order to produce the compression. The radii were carefully determined before making the apparatus by the mean of several readings with vernier callipers. The errors due to the expansion of the glass with temperature are negligible, the change in pressure caused by the expansion on heating from 20° to 80° being less than 1 part in 1,000. Having adjusted the level of the water in the indicator capillary, previously thoroughly cleaned from all traces of grease, the load was alternately applied and removed many times. It was found that freshly made bulbs could not be made to give constant readings for some weeks. Examples of this were noticed by Phillips* in some experiments on the creep effects in glass.

The decrease in the volume is measured by the rise of water in the capillary. This was ascertained by means of graduation marks on the capillary subdivided by a fine reading cathetometer and expressed primarily in cms. of capillary.

The values of the coefficient of compressibility here given have been reduced to compressibility per atmosphere.

## Compressibility of Soda Glass.

The glass employed was supplied by Mr. W. J. Nelson, of Streatham. We are indebted to Mr. E. E. Cox, of this college, for an analysis, which gave the following results :—

| | |
|---|---|
| $SiO_2$ ... ... ... ... ... ... | 66·00 per cent. |
| $Al_2O_3$ and trace of $Fe_2O_3$ ... ... ... | 4·99 ,, |
| CaO (and SrO) ... ... ... ... ... | 7·49 ,, |
| $Na_2O$ ... ... ... ... ... ... | 13·05 ,, |
| $K_2O$ ... ... ... ... ... ... | 8·30 ,, |
| | 99·83 ,, |

* Proc. Phys. Soc., 19 (1905).

Four series of experiments with this glass were carried out. The data for one of them is here given sufficiently fully to give an idea of the accuracy attained.

### Series I.

Capillary: 0·00 145 c.c.s. per cm. Load: 5,000 g. (This corresponds with $dP=5\cdot78$ atmosphere).

Radii: External, 1·325 cm. (mean of 8 readings); internal: 1·220 cm. (mean of 5 readings); internal volume: 94·6 c.c.s.

| —— | 30° | 40° | 50° | 60° | 70° | 80° |
|---|---|---|---|---|---|---|
| | mm. | mm. | mm. | mm. | mm. | mm. |
| Mean rise in capillary ... | 3·06 | 3·08 | 3·10 | 3·13 | 3·16 | 3·17 |
| Mean deviation from mean ... | 0·01 | 0·015 | 0·01 | 0·01 | 0·01 | 0·01 |

The results of the four series may be summarised as follows:—

| | | |
|---|---|---|
| Series | I. | $c=243\ \ +0\cdot2\ \ (t-30)$ |
| ,, | II. | $c=246\ \ +0\cdot16\ (t-30)$ |
| ,, | III. | $c=244\ \ +0\cdot18\ (t-30)$ |
| ,, | IV. | $c=241\ \ +0\cdot2\ \ (t-30)$ |
| Mean | | $c=243\cdot5+0\cdot185(t-30)$ |

Extrapolating to 0°C. this becomes

$$c=238+0\cdot185t,$$

a formula which should be of use to experimenters using a glass of this kind.

### Jena Glass.

Three series of measurements were made with $16^{\mathrm{III}}$ glass, but the results of the third series alone are given, as the others were not constant.

$$c=229+0\cdot28(t-30).$$

Extrapolating to 0°C. this becomes

$$c=220\cdot6+0\cdot28t.$$

### Method II.

*The application of external pressure only to a piezometer bulb.*—The relation between the decrease in volume on the application of pressure $P_1$, the elasticity constants, and the dimensions of the piezometer bulb is given by the expression

$$dV=\pi a^2 l\left\{\frac{P_0a^2-P_1b^2}{k(b^2-a^2)}+\frac{b^2(P_0-P_1)}{(b^2-a^2)n}\right\}$$

where $l$ is the length of tube,
$dV$ the change in the internal volume,
$P_0$ the internal pressure,
$P_1$ the external pressure,
$k$ the bulk modulus of the glass,
$n$ the rigidity of the glass,
and $a$, $b$, the internal and external radii respectively,*

when $$P_0=O,\ dV=\frac{\pi a^2b^2lP_1}{b^2-a^2}\left[\frac{1}{k}+\frac{1}{n}\right] \quad . \quad . \quad . \quad . \quad (1)$$

Now Poisson's ratio is given by

$$\sigma=(3k-2n)/2(3k+n) \quad . \quad . \quad . \quad . \quad (2)$$

the only assumption being that the glass is isotropic.

From (1) and (2) we get

$$\frac{5-4\sigma}{1-2\sigma}=\frac{3(b^2-a^2)dV}{c.V.b^2.P} \quad . \quad . \quad . \quad . \quad (3)$$

having put $1/c$ for $k$, where $c$ is the compressibility per atmosphere.

From equation (3) and the results of experiments by Method II the value of $\sigma$ has been derived, $c$ being known from the results of Method I.

The work of Amagat is important in this connection, since he has verified the use of these equations for glass.†

Amagat states that the diminution of $c$ with increasing pressure is within the limits of experimental error. This conclusion is of much importance, for it makes it possible to apply values of $c$ found at low pressures to the whole range of pressures used in the determination of $c$ for liquids.

## Apparatus.

The glass piezometer bulb was placed inside a steel cylinder so that a capillary tube sealed to one end of the bulb projected through the rubber packing. The bulb was filled with water, so that the level in the capillary could be read by means of a cathetometer. The whole apparatus was suspended in a thermostat after filling the remaining space in the steel cylinder with mercury, and this was connected by a steel tube to a Budenberg gauge and screw pressure apparatus filled with mercury.

The apparatus is described only briefly here, as a full description will be given when dealing with the compressibility of solutions in a later paper. The thermostat was constant to 0·01°C., and no variation (while pressure was constant) was observed in the water level in the capillary. The capillary was calibrated from an etched mark just outside the piezometer; it showed 0·02 c.c.s. per cm. Measurements

* Poynting and Thomson, Properties of Matter, p. 117 (1922).
† Comptes Rendus, 1889, 2, pp. 228 and 727.

were made at 30°, 40°, 50°, 60°, 70° and 80°C.; the numbers obtained at 80° are here given as a sample :—

Volume of bulb 25·68 c.c.s. Temp. 80°C.

External radius, 1·295 cms. Internal, 1·19 cms.

| Mean $dP$ | Mean $dV$ | Mean $\sigma$ | Mean deviation of $\sigma$ from mean value. |
|---|---|---|---|
| 13·93 | 0·01417 | 0·213 | 0·005 |

The results at the other temperatures are put together in a table at the end of this Paper.

Now $\sigma=(3k-2n)/2(3k+n)$, where $k$=the bulk modulus and $n$=the rigidity modulus. Rearranging, $n=3k(1-2\sigma)/2(1+\sigma)$. Values of $n$ have been calculated and are given in C.G.S. units.

Again, Young's modulus $q=9nk/(3k-n)$. Values of $q$ have also been calculated in C.G.S. units.

This work was undertaken originally to determine the compressibility of glass, but as it has been found possible at the same time to determine other constants with fair accuracy, it was thought worth while to publish them together. The final results are given in the following table :—

| —— | Soda glass. | Jena $16^{III}$. |
|---|---|---|
| $c$ $10^{-8}$ ... ... ... | $238{\cdot}0 + 0{\cdot}185\ t$ | $220{\cdot}6 + 0{\cdot}28\ t$ |
| $k$ $1\text{-}0^{11}$ ... ... ... | $4{\cdot}241 - 0{\cdot}00296t$ | $4{\cdot}577 - 0{\cdot}0051\ t$ |
| $\sigma$ ... ... ... ... | $0{\cdot}2207 - 0{\cdot}00012t$ | $0{\cdot}2475 - 0{\cdot}00068t$ |
| $n$ $10^{-11}$ ... ... ... | $2{\cdot}99 - 0{\cdot}002\ t$ | $3{\cdot}29 - 0{\cdot}0036\ t$ |
| $q$ $10^{-11}$ ... ... ... | $7{\cdot}27 - 0{\cdot}005\ t$ | $7{\cdot}97 - 0{\cdot}009\ t$ |

These relations have been extrapolated to 0° to make them available over the range 0° to 80°C., and may be used to 100°C. without appreciable error.

## Comparison of Results.

It will be noticed that the compressibility of soda glass is slightly greater than that of Jena glass; in each case it increases slowly with rise of temperature. The results have been converted into bulk moduli ($k$) in C.G.S. units in order to compute $n$ and $q$.

Our value for Young's modulus is here compared with the values obtained by Bell at 0°C.* for various types of glass by an acoustical method. He also observed that his results with newly worked glass were not consistent.

| Type of Glass. | $q$ $10^{-11}$ | Type of Glass. | $q$ $10^{-11}$ |
|---|---|---|---|
| Soft German ... ... ... | 6·95 | Soft Jena ... ... ... ... | 7·16 |
| Baird and Tatlock, soda ... ... | 6·91 | Soda (Nelson, London) ... ... | 7·30 |
| Powell & Sons, gauge tube ... | 7·36 | (extrapolated from our results) | |

* Bell and Chree, Proc. Phys. Soc., 19, 516 (1905).

Straubel found 0·228 for $\sigma$ in Jena $16^{III}$ at room temperature. Berkeley* gives values for the compressibility of Jena glass at 0° and 30° obtained by himself from Amagat's absolute values for the coefficient of compressibility of mercury, and his own values for the apparent compressibility. He gives also a value obtained by Straubel. The numbers are :—

| Temp. | Berkeley. | Straubel. | Perman and Urry. |
|---|---|---|---|
| 0°C. | 223 $10^{-8}$ | ... | 221 $10^{-8}$ (extrapolated) |
| 30°C. | 229 $10^{-8}$ | ... | 229 $10^{-8}$ |
| (Room temp.) | ... | 228 $10^{-8}$ | ... |

Compressibility plotted against temperature gives a straight line, so that extrapolation is easy.

No hysteresis effects were observed in any of these experiments, as the pressure was not continued a sufficient length of time.

DISCUSSION.

Mr. J. P. ANDREWS : The only value given in the Paper with which the authors' value of Poisson's Ratio for Jena $16^{III}$ glass may be compared is that of Straubel (1899), which is considerably lower. It may be pointed out that Straubel throughout his experiments, which were elaborations of Cornu's interference method, bent his glass bar over knife-edges whose maximum separation was 7 cms., the bars themselves sometimes reaching 3 cms. in width. It is known now that the separation should be greater than three times the width of the bar. For this reason, Straubel's values are certainly too small, and part of the difference between these and the authors' is to be accounted for in this way. It is, however, improbable that the whole difference is made up of this error.

* Phil. Trans., A. 218, 295 (1919).

# XXVI.—A VALVE-MAINTAINED HIGH-FREQUENCY INDUCTION FURNACE AND SOME NOTES ON THE PERFORMANCE OF INDUCTION FURNACES.

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ABSTRACT.

This paper is divided into two parts. In Part I the electrical design is given of a valve-operated high-frequency induction furnace, together with some details of its performance, and in Part II a theory of the behaviour of induction furnaces in general is developed, and some experimental results supporting the theory are given.

## PART I.—A VALVE-OPERATED HIGH-FREQUENCY INDUCTION FURNACE.

THE principles of heating by high-frequency (or ironless) induction are too widely known to render it necessary to discuss them here; it will suffice to say that the substance to be heated (which must be an electrical conductor or be contained in a conducting crucible) is surrounded by a coil in which alternating current of a suitable frequency is flowing, the eddy-currents induced in the substance serving to heat it. Several attempts have been made to propound a theory of heating by high-frequency induction, notably by Northrup,* Burch and Davis,† and Fischer.‡ The subject is referred to in Part II of this paper.

There are at least four accepted ways of producing the necessary alternating current in the inductor-coil, viz. :—

(*a*) The spark-oscillator, as used on a commercial scale by Northrup.
(*b*) The high-frequency alternator, as used extensively in Germany and in America.
(*c*) The thermionic valve oscillator.
(*d*) The Poulsen arc, which may be useful for giving high output at comparatively low frequency.

The high-frequency furnace to be described was required to be of a flexible type, capable of meeting the varied demands of a research laboratory. The valve-oscillator type lends itself admirably to the purpose, since it is possible to vary the frequency and other constants over a wide range, and also to obtain a very steady, delicately-controlled output. It may be added that the furnace was required in the first instance for the purpose of making measurements of the melting points of pure metals in a vacuum or a controlled atmosphere.

The valve-maintained induction furnace does not appear to have been used extensively on a large scale commercially, and there is practically no information published on the design and operation of such furnaces. The following notes dealing with the construction of a valve-maintained high-frequency induction furnace of 12 to 15 k.v.a. input may consequently be of interest.

* Northrup, E. F., Trans. Am. Electrochem. Soc., Vol. XXXV, p. 69 (1919).
† Burch and Davis, Phil. Mag., Vol. I, p. 768 (1926).
‡ Fischer, W., Zeits. f. techn. Phys., Vol. VII, p. 513 (1926).

The choice of a suitable frequency presents some difficulty. Theoretical investigations suggest that above a particular frequency the efficiency of an induction furnace is constant, although it falls off rapidly below this frequency. This point is dealt with more fully in Part II of this paper.

The actual value of the frequency above which the efficiency is constant is a complicated function of the geometric dimensions of the charge, its electrical resistivity and its permeability; and, in general, it is found that the critical frequency is comparatively low.

When we consider the construction of an actual furnace, however, other considerations enter. Low-frequency resonating systems demand high capacity condensers capable of passing heavy currents, and, since these are both bulky and expensive, it may be advantageous to operate on a much higher frequency, even though the efficiency may be rather less. This has been done in the furnace described in this paper, and the desired results have been obtained satisfactorily at frequencies ranging from 700 to 1,000 kilocycles per second. For the immediate purpose in hand, charges of metal not greater than about 2 kilogrammes had to be legislated for, and it transpired that the appropriate inductor coil had an inductance of about 12 microhenries. The corresponding capacity is about 0·003 or 0·004 microfarad; this can easily be obtained using comparatively inexpensive air-dielectric condensers in the oscillatory-circuit.

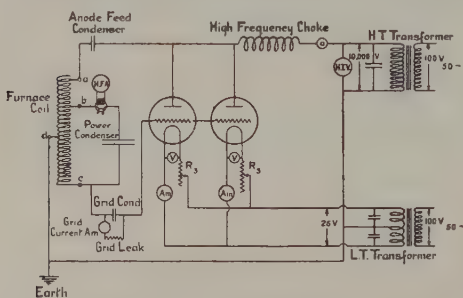


FIG. 1.—DIAGRAM OF HIGH-FREQUENCY CIRCUIT.

The choice of circuit for the valve-oscillator was governed by the following principles:—

(1) Reliability; (2) safety in operation; (3) flexibility; (4) simplicity, especially simplicity of the inductor coil, which in an induction furnace becomes the furnace coil surrounding the material to be heated.

A modification of the well-known Hartley circuit, employing a centre-tapped coil, satisfies these conditions, and was therefore adopted. Preliminary experiments were made with other circuits, but none seemed to exhibit any marked superiority over this simple circuit. The "parallel-feed" system ensures safety in operating, since the furnace coil is at earth-potential from the low frequency point of view, and the high-frequency potential developed across it is not dangerous. The high-frequency circuit used is shown diagrammatically in Fig. 1. It will be seen that two valves (of which details are given later) are used, directly in parallel, in preference to the "back-to-back" or "push-pull" system, which is in some cases more satisfactory. This procedure was adopted owing to the limitations of the power supply available. Fig. 1 is self-explanatory, but it may be useful to record the values adopted for various components. The high-frequency choke is a single layer solenoid wound on a skeleton former 12 inches in diameter, and consisting of about 250 turns of 22 S.W.G. silk-covered copper, spaced ten turns to the inch, giving an inductance of about 7,000 microhenries. The anode feed condenser, which serves to prevent the flow of low-frequency current without hindering the high-frequency current or appreciably changing the phase relations, has a capacity

of 0·002-0·004 microfarad. The grid condenser has a capacity of 0·005 microfarad, and the grid-leak, which is wire wound and oil cooled, has a resistance of 10,000 ohms. The bypass condenser across the H.T. transformer has a capacity of 0·002 microfarad, and those across the L.T. transformer are each 0·01 microfarad.

## Details of Components.

(*a*) *Valves.*—These are silica transmitting valves of the type O.C.2·5 KW. supplied by the Mullard Radio Valve Company. The following are their chief operating characteristics :—

| | | | |
|---|---|---|---|
| Rating ... ... | 3 kw. continuous anode dissipation. | Filament current ... | 40 amps. |
| | | Total emission ... ... | 3 amps. |
| | | Amplification factor ... | 62. |
| Max. anode voltage | 10,000 volts. | Impedance ... ... | 42,000 ohms. |
| Filament voltage ... | 17·2 volts. | | |

The electrode seals must be cooled by an air blast, which is supplied by blowers. In the present furnace two blowers connected to two independent supply circuits are used, and, since each blower alone is capable of supplying the necessary cooling air-blast, the valves should not be damaged by failure of one of the blowers. Signal lamps on the control panel indicate that the blowers are operating, and an electro-magnetic relay incorporated in the main circuit-breaker makes it impossible to switch on power until the blower circuits are complete, thus ensuring that the filaments of the valves cannot be heated until the air-blast is operating.

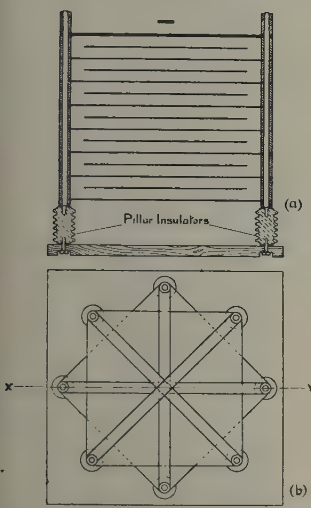


Fig. 2.—Diagram of Condensers Used in Oscillatory Circuit. Plan (*b*) and Section through *XY* (*a*).

(*b*) *High-tension Supply.*—This is supplied from a 20 K.V.A. oil-cooled transformer, and is not rectified. The high-tension voltage is regulated by means of variable resistances in series with the primary of the transformer. An alternative—though, for this purpose, a less favourable—means of control would be by a suitably designed auto-transformer or an induction regulator. The regulating resistance is in two parts. The first part is designed to carry about 20 amps., and has a total resistance of 30 ohms, thus serving to keep down the H.T. voltage to about 3,000 volts until the filament voltage is properly adjusted. This portion is then short-circuited by means of a link-switch, and the second rheostat of 2 ohms total resistance, divided among sixteen studs, serves to give the necessary fine regulation.

(*c*) *Filament Heating Current.*—This is supplied by a $1\frac{1}{2}$ K.V.A. transformer stepping down from 100 to 25 volts. Regulation is obtained partly by a rheostat included in the primary circuit of this transformer, and partly by fine adjustment rheostats in each filament circuit.

(*d*) *Condensers.*—All condensers, except those in the output oscillatory circuit, are mica-dielectric condensers. Those in the main oscillatory circuit are air-dielectric condensers designed and made at the Laboratory. Their general construction can be seen from Fig. 2. The plates (of which there are fourteen in each condenser) are of zinc sheet $\frac{1}{8}$ inch thick and 15 inches square, and the separation between the plates is 1 inch. Each condenser has a capacity of about 0·001 microfarad. Care has been taken to avoid points and sharp edges in the construction of the condensers.

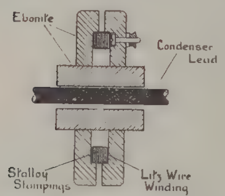


FIG. 3.—SECTION THROUGH DYE CURRENT TRANSFORMER.

(*e*) *High-frequency Ammeter.*—The various instruments used on the switch-board are standard A.C. instruments. Mention may be made, however, of the ammeter used for measuring the high-frequency current, for which purpose an ordinary hot-wire ammeter giving a full-scale deflection for 0·8 amp. is linked with the oscillatory circuit through a current transformer of the type described by Campbell and Dye.* This consists essentially of a laminated annular core of stalloy on which are wound a hundred turns of " Litz wire." The actual construction is shown in Fig. 3. The large central hole in the ebonite casing of the transformer is threaded on to one of the leads from the coil to the condenser, and the ends of the Litz winding are connected to the ammeter. The transformer thus steps down the current in the ratio of 100:1, and the ammeter has been re-calibrated to read current directly when used in conjunction with this transformer.

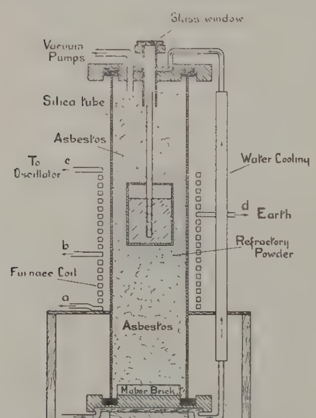


FIG. 4.—DIAGRAM OF FURNACE PROPER, SHOWING CHARGE IN POSITION AS ARRANGED FOR DETERMINATION OF MELTING POINTS.

(*f*) *The Furnace Proper.*—This is shown in Fig. 7, and also diagrammatically in Fig. 4. It consists of a silica tube round which is the inductor coil. This coil may be of solid copper strip or of flattened copper tubing wound edgewise, through which cooling water may be passed. In this furnace the coils are designed to have an inductance of about 12 microhenries. The one most frequently used (for loads of one or two kilogrammes of metal) consists of about fifteen turns of 6 inches internal diameter, space $\frac{3}{16}$ inch between turns; of these, eight turns are included between *b* and *c* (Fig. 1), and carry the main oscillatory current. The presence of the charge within the coil causes the load-impedance between the anodes of the valves and earth to be very low, as it increases the effective resistance of the oscillatory circuit. Since such a load is undesirable, it is necessary to increase the impedance of the load artificially by connecting the valves (via the feed condenser) not to the point " *b*," but

* Campbell and Dye, Proc. Roy. Soc., A, Vol. XC, p. 621 (1914); and Nat. Phys. Lab. Collected Researches, Vol. XII, p. 14 (1915); Dye, Inst. Elec. Eng. J., Vol. LXIII, p. 597 (1925).

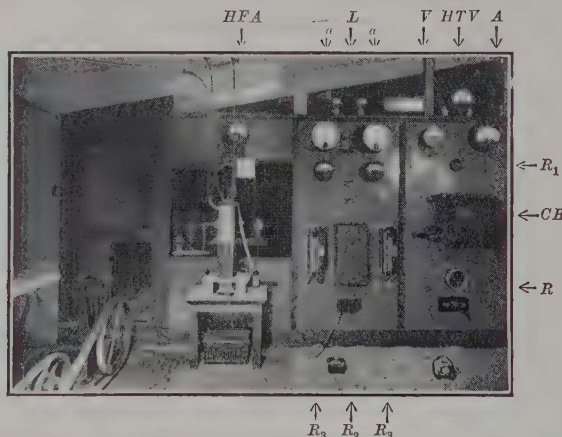


FIG. 6.—GENERAL VIEW OF SWITCHBOARD AND FURNACE.

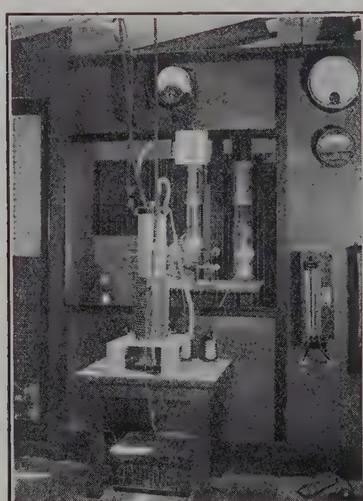

FIG. 7.—VIEW OF FURNACE SHOWING VALVES.

to some other point "*a*" on the coil. The best position of the "anode tap" (as the adjustable contact *a* is called) must be found by trial under working conditions. As the anode tap is moved away from "*b*" it is in general found that, up to a certain point, the output rises and the anode current falls; beyond this point the output falls, as also does the anode current. The best position of the anode tap is that which gives the maximum high-frequency current together with a reasonably low anode current. The tapping point *d* is approximately the mid-point of that portion of the coil between *b* and *c*. The silica tube is closed at each end by brass end pieces seating on rubber gaskets. The upper end-piece has a window for sighting into the metal, and also a tube through which the enclosure may be evacuated. The specimen to be heated is placed within the tube, as shown, being carefully positioned to be in the centre of the effective portion of the coil, and packed round with suitable lagging.

### General Lay-out.

Fig. 5 shows the arrangement of the circuits used on the low-frequency side of the equipment. To the left of the diagram is the main circuit-breaker, which controls the main circuit feeding the primary of the high-tension and filament current transformers, and is itself controlled by two subsidiary circuits. One of these subsidiary circuits must be completed before the main switch can be thrown in, and includes switches linked mechanically with the blower switches, and also a short-circuiting contact on the door to a safety cubicle in which the equipment other than the furnace proper is housed. It is thus impossible to switch on power until the door giving access to the cubicle is properly fastened and the blowers are operating. The other subsidiary circuit consists of a coil and trip connected to conveniently placed emergency pushes, and so arranged that when these are pressed all A.C. power is immediately switched off.

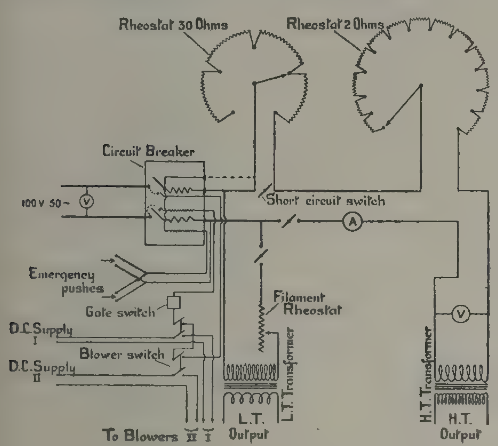


Fig. 5.—Diagram of Low-frequency Circuit.

A general view of the switchboard and control panels is given in Fig. 6. The main circuit-breaker ($CB$) is seen on the panel on the extreme right of the photograph; above the circuit-breaker are an ammeter ($A$) and a voltmeter ($V$) for measuring the power input, and above the switchboard, well out of reach, is an electrostatic voltmeter ($HTV$) giving the high-tension voltage. Immediately above the circuit-breaker is the control handle of the 30 ohm rheostat ($R_1$), the link switch for short-circuiting it being at the bottom of the panel. Just above this switch is the control handle ($R$) of the main 2-ohm rheostat.

On the adjacent panel are the instruments pertaining to the filament-heating circuits and the blowers. The centre rheostat ($R_2$) is in series with the primary of the filament-current transformer, and the two outer ones ($R_3$,$R_3$) are in series respectively with the filament of each valve. In the centre of this panel are the switches controlling the blowers and signal lamps ($L$) to indicate the behaviour of the blowers; at either side of these lamps are voltmeters and ammeters used in adjusting filament temperatures; above this panel are ammeters ($a$, $a$) for measuring the D.C. component of the anode current through the valves.

Further to the left is a wire grid, through which the valves, etc., in the cubicle may be observed. This portion is shown more clearly in Fig. 7. In front of this grid is the furnace itself, mounted on a table. Above the wire grid is the ammeter ($HFA$) for measuring the high-frequency current in the furnace coil, and just below it is a U-gauge to indicate the pressure within the silica tube.

### Operation and Performance.

Given a particular crucible of metal to be melted, one must first select an inductor-coil of suitable size for efficient heating. For small charges not exceeding one or two kilogrammes, experience has shown that the radius of the coil should usually be about twice that of the crucible; complications occur, however, in certain instances, especially in the case of relatively poor conductors (such as carbon and ferromagnetic materials), since loads of this nature impose a heavy damping on the oscillator. To some extent the undesirable effects of this very low load-impedance may be compensated by raising the position of the anode tap "$a$" (Fig. 1); but it is generally necessary to use an inductor coil of considerably greater diameter in order to obtain stable oscillations. The inductance of the coil should be made approximately the correct value to suit the oscillator, this value being decided chiefly by the frequency and the oscillatory current to be passed at a given high-tension voltage.

If a coil of small diameter is required, it usually happens that its ohmic resistance is unduly large, since it is necessary to have a large number of turns, and as it is desirable to make the ohmic resistance as low as possible, another arrangement must be adopted. The total inductance associated with the oscillatory circuit is divided between two separate coils, one of which has the radius and length necessary to give efficient heating, and the second being designed to have minimum high-frequency resistance (i.e., its diameter is about 1·5 times its length). The two coils are then connected in series, and the crucible inserted in the appropriate one.

The furnace has proved capable of meeting the demands of an investigation dealing with high melting points. For example: Kilogramme charges of palladium (M.P. about 1550°C.), nickel (M.P. about 1450°), chromium (M.P. about 1630°), and iron (M.P. about 1530°) have each been melted successfully, using about 8½ kilowatts input.

## Part II.—Performance of High-frequency Induction Furnaces.

*Introduction.*

Several attempts have been made to analyse mathematically the problem of heating by high-frequency induction since the original account of the furnace was published by Northrup,* and, as has been pointed out by Burch and Davis, some

* Northrup, Trans. Am. Electrochem. Soc., Vol. XXXV, p. 69 (1919).

of these analyses are not reliable. Burch and Davis* have themselves published a fairly rigid solution to the problem for the case in which the charge consists of a solid cylinder, and the inductor consists of a single cylindrical sheet, long compared with its radius, and complete except for an infinitesimal longitudinal gap across which the condenser is connected.

The " efficiency " of a furnace may be defined as the ratio of the energy appearing as heat in the charge to the total energy in the whole system of inductor and charge. Thus, if $W_C$ watts appear as heat in the charge, and $W_H$ watts are lost in heating the coil, and $W_R$ watts are lost by radiation, the efficiency of the furnace may be defined as

$$\eta_1 = W_C/(W_C + W_H + W_R)$$

This definition of efficiency takes no account of the energy inherently lost in whatever device is used for producing the necessary high-frequency current. If $\eta_2$ is the efficiency of conversion from low-frequency current to high-frequency current, the quantity $\eta_1\eta_2$ represents the ratio of energy appearing as heat in the charge to the total energy supplied to the entire system, and may be defined as the " over-all efficiency."

In the definition of $\eta_1$ given above it has been assumed that the losses in the synchronising condensers are negligibly small, and it can easily be shown that the loss by radiation ($W_R$) is also small compared with $W_C$ and $W_H$.

The efficiency of the furnace is a function of the following variables :—

Radius of coil ($R_1$) ; radius of charge ($R$) ; resistivity of charge material ($\rho$) ; resistivity of coil material ($\rho'$) ; permeability of charge material ($\mu$) ; permeability of coil material (assumed to be unity) ; frequency ($f$) ; length of charge ($l$) ; and to determine the thermal input to the charge we shall also require the R.M.S. value of the high-frequency current $I$.

The following parameters also occur in the discussion

$$\beta' = \sqrt{\rho/4\pi i p \mu} \quad \text{(used by Burch and Davis)}$$

and

$$m = \sqrt{4\pi p \mu/\rho} = \sqrt{8\pi^2 \mu f/\rho}$$

where $p$ is the pulsatance in radians per second and $i = \sqrt{-1}$. In their analysis of the efficiency of a high-frequency induction furnace constructed as already explained, Burch and Davis find that the efficiency is independent of frequency, provided that $R/\beta' > 3$, and has a value

$$\eta_1 = lR\sqrt{\rho}/(lR\sqrt{\rho} + l'R_1\sqrt{\rho'})$$

if $\mu$ is assumed to be unity.

A similar result is obtained as a result of the following discussion, which, although not claiming the mathematical rigidity of the solution of Burch and Davis, possibly brings out more clearly the physical processes involved, and, by taking approximate values for the resistance of a coil at high frequency, corresponds perhaps a little more closely to the real case.

* Burch and Davis, Phil. Mag., Vol. I, p. 768 (1926).

Thus the determination of $\eta_1$ conveniently resolves itself into a determination of (*a*) the heat generated in the charge, and (*b*) the heat generated in the inductor coil as a result of its resistance.

The problem of determining the heat developed in the charge is analogous to that of determining the losses by eddy currents in a transformer core, which problem has been solved by Heaviside* and by Russell† for the case in which the coil is indefinitely long, so that the value of the field between the coil and the boundary of the charge has everywhere a value of $4\pi nI$ (where $n$ is the number of turns per unit length in the coil) and the alternating current varies according to a harmonic law. Using the method adopted by the latter, the core loss (which is, in this case, the heat generated in the specimen) is found to be

$$W_c = \frac{\pi R l f \mu H^2}{m} \cdot \frac{Z(mR)}{X(mR)} \cdot 10^{-7} \text{ watts} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $H$ is the R.M.S. value of the magnetic intensity at the boundary of the core, $Z(mR) = ber(mR)ber'(mR) + bei(mR)bei'(mR)$ and $X(mR) = ber^2(mR) + bei^2(mR)$.

If $(mR)$ is sufficiently great, $Z(mR)/X(mR)$ approaches the limit $1/\sqrt{2}$, the exact value being only slightly less if $mR > 10$. When this value is taken we have

$$W_c = \tfrac{1}{4} \cdot RlH^2\sqrt{\mu \rho f} \cdot 10^{-7} \text{ watts}$$

$$= 4\pi^2 R l n^2 I^2 \sqrt{\mu \rho f} \cdot 10^{-7} \text{ watts} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

These expressions give the heat generated by eddy currents in the charge if the coil is indefinitely long. To the general accuracy contemplated (say 10-15 per cent.) it can be shown that the field within the coil is sufficiently nearly uniform and of value $4\pi nI$ if the coil is only slightly longer than its diameter. Hence the energy generated in the charge can be approximately calculated for most practical cases. It will be shown later in this Paper that the agreement between this calculated value and an observed one is sufficiently good. The length of the charge must clearly not be so great that part of it lies in a field of intensity widely different from that given by $4\pi nI$.

(*b*) The coil loss may be calculated approximately from the values given by Howe‡ for the high-frequency resistance of coils wound of wire of square or rectangular section. The corresponding problem for coils wound with wire of circular cross-section has been treated by Butterworth.§ As the coils usually employed in induction furnaces are wound with wire of square or rectangular section, only this type is considered in detail. The relation between low-frequency resistance ($R_0$) and high-frequency resistance ($R_c$) for such coils is

$$R_c = R_0 \beta \tau / \gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $\tau$ is the radial thickness of the wire, $\beta = 2\pi\sqrt{f\mu/\rho}$, and $\gamma = (\cosh 2\beta\tau - \cos 2\beta\tau)/(\sinh 2\beta\tau - \sin 2\beta\tau)$.

* Heaviside, Electrican (1884), and Collected Researches.
† Russell, Alternating Current, Vol. I, p. 503.
‡ Howe, Proc. I.E.E., Vol. LVIII, p. 152 (1920).
§ Butterworth, Wireless World, Vol. XIX, p. 754 and 811 (1926).

Fig. 8 (reproduced from the paper by Howe) shows the relation between $R_c/R_0$ and $\beta\tau$. It will be seen that if $\beta\tau$ is greater than about 2, $R_c/R_0$ becomes very nearly directly proportional to $\beta\tau$, that is, other things being equal, directly proportional to the square root of the frequency.

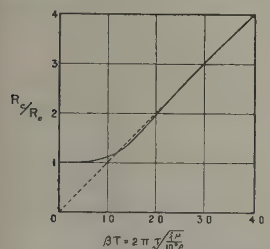

FIG. 8.—CURVE (AFTER HOWE) SHOWING RELATION BETWEEN $R_c/R_0$ AND $\beta\tau$, FROM WHICH $R_c$ MAY BE FOUND FOR ANY VALUE OF $\beta\tau$.

The expression $R_c = R_0\beta\tau/\gamma$ holds for closely-wound, indefinitely long coils: if the turns are slightly separated, the flux density will be reduced. The same result would be brought about by a decrease of permeability from unity to $n\tau_1$, where $\tau_1$ is the axial thickness of the wire. Further, the effect of the finite length of the coil may be allowed for by taking a further reduced value of the flux density. For a coil whose length is greater than its diameter, this correction does not amount to more than a few per cent.

Thus spaced winding and finite length can be allowed for by taking, instead of $\beta\tau$, a value $\beta\tau\sqrt{n\tau_1} . \sqrt{B_1/B}$. The correction factor $\sqrt{B_1/B}$ may be determined from the following values of $H_1/H$ given by Howe. (Loc. cit.).

| | | | |
|---|---|---|---|
| $l'/2R_1$ | 1·0 | 2·0 | 4·0 |
| $H_1/H$ | 0·91 | 0·96 | 0·98 |

It is thus possible to determine the behaviour of the furnace under any given conditions to an accuracy sufficient for most purposes, when the charge consists of a solid right circular cylinder of conducting material placed coaxially within the inductor.

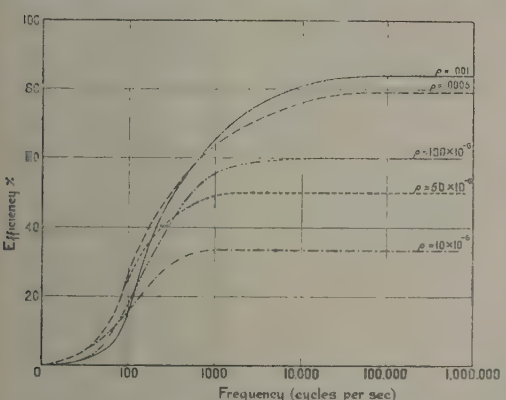


FIG. 9.—VARIATION OF EFFICIENCY WITH FREQUENCY FOR DIFFERENT VALUES OF RESISTIVITY (ρ) FOR THE PARTICULAR CASE OF A CHARGE OF RADIUS 5 CMS. AND LENGTH 10 CMS. WITHIN AN INDUCTOR OF RADIUS 10 CMS. AND LENGTH 20 CMS.

It will be seen from equations (2) and (3) that the heat generated in the cylindrical charge, and also the energy lost in heating the coil, both become proportional to the square root of the frequency after the frequency has attained a certain value. This shows that, beyond this "critical frequency," the efficiency of the furnace is independent of frequency.

Fig. 9 shows graphically the variation of efficiency with frequency for different values of the resistivity in the case of a cylindrical charge of radius 5 cms. and length 10 cms. within a coil of radius 10 cms., length 20 cms., made of 20 turns copper strip 0·5 cm. square in section.

## Experimental Results.

An investigation into the behaviour of induction furnaces was undertaken with the aid of the valve-maintained furnace described in Part I of this paper.

In this investigation the energy supplied as heat to the charge and that lost by heating the inductor were separately determined.

The inductor consisted of a water-cooled coil 7 cms. radius of 10 turns spaced 1·2 turns per centimetre, constructed of copper tubing of circular section $\frac{1}{4}$ diameter externally.

The inductor fitted closely round a silica tube, and within this was a second non-conducting refractory tube, packed round with asbestos wool, as shown in Fig. 10. The object of this second tube was to permit the specimen under investigation to be inserted and withdrawn without disturbing the lagging.

In making a determination of $W_c$ and $W_H$ the specimen was put in the inductor and the high-frequency current maintained at a constant value. Under these conditions the charge attained a certain steady temperature, at which the heat supplied by the furnace was equal to that lost to the surroundings. This temperature was measured by means of an optical pyrometer, and subsequently used to determine the energy input to the charge. The energy lost in heating the coil was determined by noting the temperature of the cooling water on entering and leaving the coil and measuring the flow. This was done during the early stages of the experiment, before any appreciable heat reached the coil from the charge by conduction through the lagging. By this means the coil resistance was found to be 0·34 ohm, as a mean of many observations under different conditions.

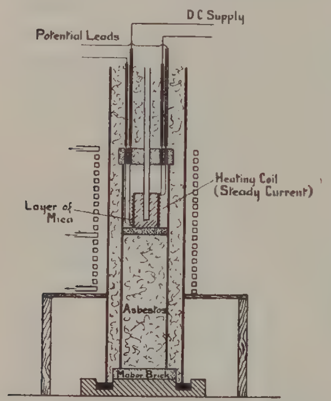


Fig. 10.—Diagram of Furnace arranged for Making Observations on the Performance of Induction Furnaces.

After the charge had attained a final steady temperature, the power was switched off and, after cooling sufficiently, the charge was withdrawn carefully, so as not to disturb the lagging, and wrapped with a heating coil of nichrome wire, separated from the specimen by the thinnest possible layer of mica for insulation purposes. The specimen was then carefully replaced within the coil, and a steady direct current passed through the heating coil, so as to raise the specimen to the same final temperature as with the induction furnace. It was assumed that the energy necessary to raise the specimen to the steady temperature was approximately the same as that supplied by the furnace. This seems probable, since heating takes place primarily at the surface in each case, and the lagging which determines the heat loss is the same in each case. Actually it was found necessary to ascertain the temperature corresponding to values of the energy supplied by the heating coil, and determine the energy developed in the charge when heated by the induction furnace from the curve representing these observations.

The results of these observations for various specimens at a frequency of $10^6$ cycles per sec. are given in Table I. The energy is given as watts per amp. current in the coil.

TABLE I.

| Material. | Radius. | Length. | Resistivity. | Energy calculated. | Energy observed. |
|---|---|---|---|---|---|
| Nickel cylinder ... | 2·8 cms. | 7·5 cms. | $4 \times 10^{-5}$ | $15{\cdot}2 \times 10^{-2}$ | $16{\cdot}6 \times 10^{-2}$ |
| " " ... | 2·5 " | 3·8 " | $4 \times 10^{-5}$ | $7{\cdot}5 \times 10^{-2}$ | $8{\cdot}0 \times 10^{-2}$ |
| Graphite cylinder | 2·5 " | 5·7 " | $1{\cdot}5 \times 10^{-3}$ | $58{\cdot}6 \times 10^{-2}$ | $47{\cdot}4 \times 10^{-2}$ |
| " " ... | 4·0 " | 6·0 " | $1{\cdot}5 \times 10^{-3}$ | $105 \times 10^{-2}$ | $96{\cdot}0 \times 10^{-2}$ |

The value of the resistivity given in the fourth column is that at the final temperature which was adjusted to 850-1,000°C.

The agreement between observed and calculated values is better than would be expected from the nature of the problem. The fact that we are dealing with coils of finite length and charges whose length is comparable with that of the coil would lead one to expect the observed values to be considerably less than the calculated ones. That they are not less is possibly due to the presence of an appreciable amount of energy in the higher harmonics of the high-frequency current. It does appear, however, that the values calculated from the simple expression given above are of the right order.

The charges used in high-frequency furnaces are usually in the form of small fragments contained in a suitable crucible. Although it appears impossible to calculate the probable heat developed in such a charge, observations were taken on a crucible of nickel shot and one on a crucible of fragments of chromium, with the results given in Table II.

TABLE II.

| Material. | Radius (of crucible). | Length (of charge). | Resistivity. | Energy observed (watts per amp.). |
|---|---|---|---|---|
| Nickel shot ... ... | 2·5 cms. | 5 cms. | $4 \times 10^{-5}$ | $49{\cdot}0 \times 10^{-2}$ |
| Chromium pieces ... | 2·5 " | 5 " | $15 \times 10^{-5}$ | $67{\cdot}0 \times 10^{-2}$ |

The energy generated in these cases is not, as might have been expected, proportional to the square root of the resistivity, but while the specimen of nickel was in the form of shot of mean diameter, about 4-5 mms., the chromium was in the form of irregularly shaped fragments of varying sizes, and this probably accounts for the disparity.

The Author desires to acknowledge his indebtedness to the Superintendent of the Physics Department (Dr. G. W. C. Kaye, O.B.E., M.A., D.Sc.), at whose instigation the work was begun, and whose continued interest and assistance have rendered its conclusion possible, and to thank Mr. W. F. Higgins, M.Sc., for advice, and Mr. R. R. Strand, A.M.I.E.E., for advice and help.

## DISCUSSION.

Mr. B. S. GOSLING said that the paper gave exactly the information which was required by those concerned. Referring to the high-frequency generating circuit shown in Fig. 1, he suggested that a push-pull arrangement might be preferable to the parallel pair of valves indicated, since transformers are inclined to perform badly with unidirectional current. It might be useful to note that stalloy stampings are not essential to the Dye type of transformer shown in Fig. 7, which will work quite well with an air core. Had the author considered the

possibility of X-radiation from the apparatus, which is intended for use by non-specialists? The speaker had recently found, in testing a commercial apparatus employing valves of large power, that at a particular adjustment the anode voltage rose to double the high-tension supply. In the present case it was conceivable that X-radiation equivalent in volume to 10 mg. of radium and sufficiently hard to be dangerous, in the absence of protective screening, might be produced.

Dr. R. T. BEATTY noted with interest the author's conclusion that high frequencies, permitting the use of small condensers, are to be preferred in induction furnaces. The paper should have considerable influence on commercial practice. On what basis had the inductor coil mentioned in the penultimate paragraph of Part I been designed? It had been shown that with a prescribed diameter the optimum length of an inductance solenoid is infinite, with a prescribed volume it is zero, and with a prescribed surface it is, according to Butterworth, half the diameter, whereas in the present case the length was two-thirds of the diameter.

Dr. W. H. ECCLES said that the Author had done valuable work in attacking the problem of induction-furnace design from first principles, since high-frequency apparatus has hitherto been designed for wireless purposes, which require high voltages, whereas the present apparatus requires large currents. For instance, the Poulsen arc as designed by wireless engineers might be quite unsuited for the present purpose. What were the relative efficiencies of the Northrupp furnace and the present furnace? Valve apparatus seemed more suitable than high-frequency alternators for such furnaces, since the variation in load produced by the melting of the charge affects the circuit constants of the valve system so as to cause some degree of self-adjustment. Had the author found a change of frequency corresponding to such a change in load? The conclusion that above 10,000 cycles the frequency is immaterial to efficiency was an important one.

Dr. H. D. H. DRANE (communicated): The author's treatment of the essentials of furnace design and operation in Part I is very clearly put, and useful from a practical standpoint. I note that the arrangement shown in Fig. 1 was adopted "owing to limitations of the power supply available." Since the valves take only a half cycle supply, this arrangement would seem likely further to cramp the existing limited power supply. The back to back arrangement would not only evade the objectionable feature of half-cyclic working, but also halve the amplitude of the current supply pulses experienced with the arrangement of Fig. 1. It cannot be doubted that with industrial conditions of operation, power supply companies would object to the non-cyclic operation which the Author has adopted. A high-frequency furnace which I have had in use satisfactorily for about three years for experimental work upon ceramic materials overcomes these and other difficulties. In this two 10 kw. (anode dissipation) silica valves are arranged so that each works upon its appropriate half of the supply current cycle, and the demand upon the public power supply is thus regularised. The high-frequency side is correspondingly improved, since a sustained wave train is obtained, in place of intermittent wave trains sandwiched between idle half cycles given by the arrangement of Fig. 1. The Author would seem to sacrifice some efficiency and a measure of satisfactory operation by using the simple expedient of a grid leak and condenser only. For good and foolproof operation a permanent grid potential should be applied, and in the much larger generator to which I have referred above this is arranged by means of a specially designed grid transformer in conjunction with the grid network. With a rectified anode voltage supply this generator would give about 80 kw. output, though as at present arranged its maximum output is 15 kw.

The production of $X$-radiation from the anodes of valves during operation for furnace work, mentioned by Mr. Gosling, is not a risk to the operator under industrial conditions. With peak voltages of the order 15-20 kv. the $X$-radiation produced is not very penetrating; and with either a silica valve or with a water-cooled anode type valve, the $X$-radiation is emitted, almost exclusively, axially from the anode cylinder, which, with usual valve mountings, means away from the operator.

AUTHOR'S reply: With reference to the remarks of Mr. Gosling and Dr. Drane concerning the alternative "push pull" arrangement, I freely admit that there is every reason to believe that, in general, this would be more satisfactory from many points of view. The furnace just described, although a permanent addition to the range of furnaces in use in the department, is erected in a temporary building, in which the voltage of the available A.C. supply is far below normal; consequently it is impossible, with the existing apparatus, to obtain the 18,000 to 20,000 volts necessary for symmetrical working. The voltage could, of course, be raised initially by using a low-ratio step-up transformer; but as the furnace fulfils its immediate requirements, this step is not necessary at the moment. The modifications necessary for symmetrical working

are very slight, and the values of the various capacities, inductances, etc., would probably not be appreciably altered.

No experiments have been made with regard to the production of X-rays from the valves, but tests will be made at the next opportunity. The only protection from X-rays is about 7 ft. of air, which is much more than the minimum given by the makers of the valves. As a result of the discussion following a series of Papers on various kinds of valves read before the I.E.E (*see* Inst. El. Eng. J., 65, pp. 812-822, 1927), it was considered that there was not likely to be any danger from this source. The experience of Mr. Gosling shows very clearly that no apparatus should be considered as safe until it has actually been proved to be safe by suitable measurements.

With regard to the question about coils asked by Dr. Beatty, the problem arises when it is required to heat a very small crucible of metal—say, three quarters of an inch in diameter and two inches long. The coil must fit fairly closely round the crucible, and hence must have a diameter of only about one or one and a half inches ; further, its inductance must be of the order of 12 microhenries. To obtain this inductance in a single layer coil a large number of turns is required, and the coil is unwieldy, as well as having an unnecessarily high ohmic resistance. Consequently, it is advantageous to make a coil of convenient shape and reasonably low ohmic resistance to give the required inductance, and to put this in series with the small coil of only a very few turns which serves to heat the metal. To meet the case of heating a small charge, Northrup devised what he calls the " focus inductor," which is described in his Paper in the Trans. Am. Electro-Chem. Soc. already mentioned. Efforts to utilise this in the valve-maintained furnace have not been successful.

Although no definite information is available, there is reason to believe that the efficiencies of the valve furnace and the Northrup furnace are comparable.

The interesting point regarding the change of frequency on melting also raised by the President will be investigated as soon as possible ; but I think that little change will be found when the charge consists of a solid cylindrical ingot. It is a suprising fact that there is, in general, no appreciable change in the operating constants (output current, etc.) in these circumstances. On melting, the shape of the ingot does not alter, and hence it would appear that there can be no abrupt change of resistivity on passing from the solid to the liquid state.

On melting a crucible full of pieces of metal a very definite increase of oscillatory current occurs on melting, and presumably there is a correspondingly great change of frequency.

# XXVII.—THE AMPLITUDE OF SOUND WAVES IN RESONATORS.

By E. G. Richardson, *B.A., D.Sc., Ph.D.*, University College, London.

*Received March* 24, 1928.

Abstract.

After a consideration of the relations between the "pipe" and the "Helmholtz resonator," graphs of the variation of amplitude, obtained by means of a hot-wire anemometer traversed through various types of resonator, are reproduced. By means of a calibrated manometer of the vibrating membrane type, some direct values of the impedance of orifices are obtained. With the hot wire, traverses across an orifice through which the air is vibrating in simple harmonic form are obtained, which show a tendency for the air to vibrate with greater amplitude in annuli remote from the centre of the orifice.

## I. Relation between Pipe and Helmholtz Resonator.

In an earlier paper,* measurements of velocity and displacement amplitude at a number of points along a sounding organ-pipe were described. The present paper describes a continuation of the former work by similar methods to the study of the vibration in a sounding Helmholtz resonator. Interest in these two types of vibrating bodies of air seems to have been awakened lately, on the one side stimulated by the invention of the Tucker hot-wire microphone, and on the other evidenced by the results of several new measurements on the end-correction of pipes, published during the last few months, to be mentioned later.

The motion in the theoretical pipe consists of stationary waves; its longest dimension is comparable with the wave-length, and the displacement along the interior rises or falls continuously. Whereas in the more sensitive type of resonator—first devised by Sondhauss, but usually named after Helmholtz—we have a cavity of air whose dimensions are small in relation to the wave-length, and which communicates with the atmosphere by an orifice or neck whose cross-section is so restricted compared to that of the cavity that, in theory at any rate, the to-and-fro motion of the air may be regarded as confined to its neighbourhood. The classical theories of the "pipe" and the "resonator" are therefore mutually exclusive. In the former, changes of cross-section are excluded, the variation of displacement with distance along the pipe being all due to progressive phase-angle; in the latter, spatial phase-differences are excluded, the variation in displacement amplitude being due to sudden changes in cross-sectional area. In consequence of this, the law connecting the resonant frequency and the length of a cylindrical "stopped pipe" and the law for a Helmholtz resonator consisting of a cylindrical tube closed by a narrower hole at one end, are fundamentally different.

The formula for the pipe (lowest tone) is

$$4nL=c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

($n$ is the frequency, $L$ the "corrected" length, and $c$ the velocity of sound).

* Proc. Roy. Soc., 112, 521 (1926).

For the resonator,

$$\omega = 2\pi n = c(\kappa/v)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

($\kappa$ is the " conductivity " of the orifice, $v$ the volume of the cylindrical cavity $SL$, when $S$ is the cross-sectional area of cylinder and $L$ its length.)

Then (1) gives $nL$ constant, whereas (2) gives $n^2L$ constant. Both these formulæ have been tested and found to fit the extreme conditions fairly closely, provided that with the value of $L$ for the " pipe " is incorporated an " end-correction " amounting to 0·58 times the radius of the pipe, but few experiments seem to have been made on intermediate cases with a view to determine how a compromise is effected between the two end conditions and how one law passes over into the other.

By adapting the idea of acoustic " impedance " first introduced by Webster* it is possible, however, to obtain a formula (soluble by graphic methods) to cover all cases, as follows :—

If the displacement $\xi$ of the air is uniform over any one section (area $S$)—i.e., if viscous resistance can be neglected—the velocity produced by an applied pressure difference $p$ can be written in the form $S^{-1}(dX/dt)$, where $X$ is of the dimensions $[L^3]$, and is a species of volume displacement. The impedance is defined as the ratio of these two quantities :—

$$Z = p/\dot{X} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

As $p$ is alternating ($=Pe^{i\omega t}$), its average value $P$ is taken in calculating $Z$.

Taking $p = Pe^{i\omega t} \cos(\omega L/c)$, with the further equation: $\rho\delta^2\xi/\delta t^2 = -\delta p/\delta x$ we obtain, for the volume displacement in a " stopped pipe "

$$\dot{X} = -(SPe^{i\omega t}/i\rho c)\sin(\omega L/c),$$

the effective impedance of the " pipe " becomes

$$Z = p/\dot{X} = -\rho c/S \tan(\omega L/c) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The equation of the motion in the orifice of the air of mass $m$ is

$$(m/A^2)\ddot{X} = Pe^{i\omega t}.$$

Integrate, and $(m/A^2)\dot{X} = (P/i\omega)e^{i\omega t}$, whence $Z_0 = \omega m/A^2$. But $m = Al\rho$, where $l$ is the effective length of the orifice in the direction of vibration, and $A$ is its area of cross-section. Put $A/l = \kappa$, then $Z_0 = \omega\rho/\kappa$.

To get the total impedance of a pipe having such an orifice, the component impedances must be added in series—i.e.,

$$Z_1 = Z_0 + Z = \omega\rho/\kappa - \rho c/S \tan(\omega L/c) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Resonance will occur when $Z_1 = 0$ or a minimum.

Strictly speaking, a viscous resistance $F$ should be included in $Z_1$, preventing $Z_1$ from ever reaching zero in practice. $F$ corresponds to " ohmic resistance " in electrical cases, and will be considered later. There is, in addition, the " radiation resistance " of the orifice to be considered if the resonator is designed to radiate sound ; but as this is a function of $(\omega/c)^2$, and $(\omega/c)$ is a small quantity, it may be neglected in these experiments.

* Nat. Acad. Sci. Proc., 5, 275 (1919).

Otherwise, if $Z_1=0$, then

$$\tan (\omega L/c)=\kappa c/\omega S$$

or putting

$$\omega/c=k$$

$$\tan kL=\kappa/kS \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

For the ideal pipe, the conductivity of the orifice is infinite, giving $\tan kL=\infty$, which gives $nL$=constant. When, however, $\kappa$ is comparatively small, so that $\tan kL=kL_3$ nearly, we have the resonator formula $n^2L$=constant. Our equation (6) thus embraces all cases, and the natural frequency may be found in any individual case by plotting $\tan kL$ against $(kL)^{-1}$, and finding where this curve is cut by the straight line, obtained by plotting $\kappa/kS$ against the same abscissa, or, alternatively, by the use of tangent tables.

It may be noted in passing that when the impedance $Z_0$ consists of an orifice of equal, or nearly equal, cross-section to that of the pipe itself, the difference in the value of $L$ from (6) corresponding to the appropriate value of $\kappa$ and $k$, and that value given by $\tan kl=\infty$ is the end correction of the pipe.*

In order to test this formula, some resonance-tube measurements have been made in the usual fashion, one end of the tube being terminated by a water surface and the other covered with a thin metal plate pierced with a cylindrical hole. The fundamental length which resounded to various tuning forks for orifices of various diameters were determined, and are exhibited in the following table. I am indebted to Miss W. Curtis, a student, for assistance in obtaining these results.

TABLE I.

First tube: 4·3 cm. diam.

| Diam. of orifice (cm.) | Resonant lengths (cm.). | | | | |
|---|---|---|---|---|---|
| | $n=$ 1,000 | $n=$ 500 | $n=$ 335 | $n=$ 246 | $n=$ 224 |
| 0·70 | 1·7 | 2·9 | 11·5 | 17·0 | 20·5 |
| 1·15 | 2·7 | 9·0 | 15·0 | 25·0 | 29·8 |
| 1·6 | 3·5 | 11·0 | 18·2 | 28·0 | 32·0 |
| 2·2 | 4·0 | 12·5 | 20·0 | 29·7 | 34·2 |
| 2·6 | 4·7 | 13·2 | 21·0 | 30·0 | 35·4 |
| 3·2 | 5·3 | 14·0 | 22·2 | 31·4 | 36·0 |
| 4·3 | 6·0 | 15·4 | 23·6 | 32·5 | 36·8 |

Second tube: 2·2 cm. diam.

| Diam. of orifice (cm.) | Resonant lengths (cm.). | | | | |
|---|---|---|---|---|---|
| | $n=$ 1,000 | $n=$ 500 | $n=$ 335 | $n=$ 246 | $n=$ 224 |
| 0·7 | 4·2 | 11·5 | 19·0 | 27·8 | 32·1 |
| 1·2 | 5·2 | 14·5 | 21·5 | 30·2 | 35·0 |
| 1·6 | 5·9 | 15·0 | 22·6 | 32·0 | 35·6 |
| 1·9 | 6·2 | 15·0 | 22·8 | 32·5 | 36·3 |
| 2·2 | 7·0 | 16·0 | 24·4 | 32·8 | 37·8 |

* The value of the end-correction of a pipe has been worked out in connection with the stationary wave method for measuring absorption coefficients from impedance principles in a process differing from the above by E. T. Paris, Phil. Mag., 4, 907 (1927).

Taking the conductivity of a simple circular orifice to be equal to its diameter (nearly) Figs. 1 and 2 show the variation of $L$ with $\kappa$ for the various frequencies, together with the theoretical values from equation (6). If the frequency be kept constant, $\tan kL \propto \kappa/S \propto r/R^2$, where $r$=radius of the orifice and $R$ of the tube. The theoretical values were obtained by calculating $\tan^{-1}(\kappa/Sk)$ in radians, giving values of $kL$, and hence of $L$. It can be seen that the theoretical and the experimental curves are similar, but that there is a lack of agreement when $r$ is large. This, one may judge, is due to a change in the dependence of $\kappa$ upon $r$. It is evident that some such change must take place as the "resonator" conditions are approached for then the effective length of the orifice is changed in virtue of the large pressure gradient through the orifice. It seemed better, however, to plot these graphs on the assumption that the conductivity remained equal to the diameter.

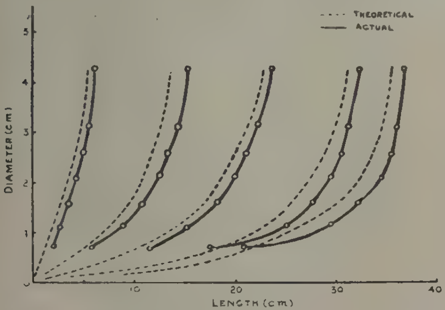


FIG. 1.—RESONANT LENGTHS: TUBE 4·3 CM. DIAMETER.

The discrepancy is greater with the wider tube, which confirms what Anderson and Ostensen* note—i.e., that the end-correction increases relatively in wider pipes. Irons† has found, in his experiments on constrictions in Kundt's tube, that the resonator formula (2) holds good if the orifice is less than 1 cm. in diameter. The present results indicate a limit of this order, but the range of orifice for which (2) holds good depends also on the frequency.

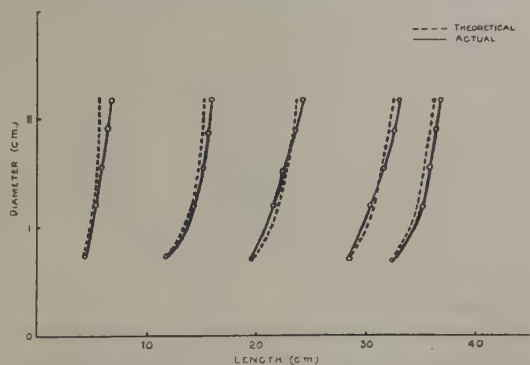


FIG. 2.—RESONANT LENGTHS: TUBE 2·2 CM. DIAMETER.

## II. Displacement Amplitude through Helmholtz Resonators.

In view of the good agreement between theory and practice in regard to the dimensions of Helmholtz resonators and their resonant frequency when the dimensions of the orifice were small compared with those of the cavity, it was expected that the assumption of Helmholtz that the motion is confined to the neck would be nearly justified. In order to test this, traverses were made from orifice to rear of Helmholtz resonators of various sizes and shapes, with the hot-wire to give measure-

* Phys. Rev., 31, 267 (1928).
† Phil. Mag., 5, 580 (1928).

ments of the displacement amplitude in the same way as is described in the former paper for the pipe.

The essence of this method is that the steady fall of resistance of a short length of 0·001 in. diameter wire (of platinum in some cases, of nickel in later measurements) when cooled by the oscillating air-draught, is measured on a form of Kelvin bridge. The wire is calibrated in the steady draught of a small wind-channel worked at known wind velocities. It has been shown that the change of resistance in the simple harmonic draught is the same as would be produced by a steady draught at velocity equal to the maximum in the alternation. Hence the amplitude of the aerial motion can be calculated.

Fig. 3 shows the variation of amplitude through a cylindrical "bottle" resonator (capacity 6,000 c.c., neck 3 cm. diam. by 7 cm., and frequency 75·5) excited by a tuning fork whose prongs vibrated 1 cm. in front of the mouth, in a plane perpendicular to the plane of traverse, at two different excitation amplitudes.

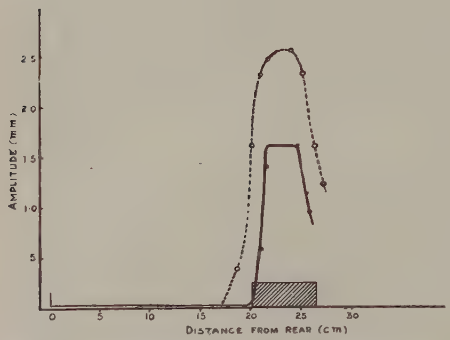


FIG. 3.—AMPLITUDE IN "BOTTLE" RESONATOR.

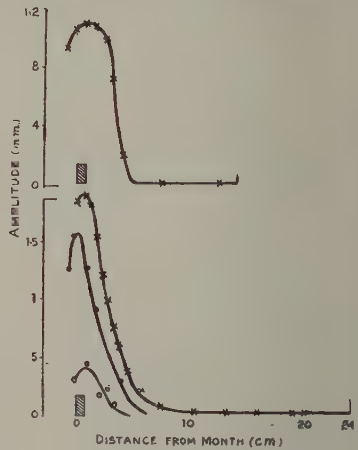


FIG. 4.—AMPLITUDE IN RESONATORS WITH CIRCULAR-HOLE ORIFICES.

The hot wire consisted of 2 cm. of 0·001 platinum stretched between the ends of two thin brass rods, which could be pushed from outside through the mouth to the rear wall of the resonator. The extent of the "neck" is shown by the shaded portion of the figures.

Fig. 4 shows similar results for two resonators of spherical type, having a circular hole as "mouth." These resonators were provided with nipples at the back for attachment to manometric capsules, enabling the pressure amplitude in the interior to be measured at the same time as the velocity in the mouth. For this purpose the manometric capsule membrane shown in Fig. 5 was employed.

The capsule widens out conically from a short cylindrical piece to a flange between which and a corresponding ring there is fixed by screws a membrane of the thinnest sheet rubber, tightly stretched in order that the natural frequency may be high. A mirror formed by silvering a tiny fragment, about 1 mm. square, of glass from a cover slip was cemented by rubber solution at a distance equal to

half the radius, from the centre of the membrane. On this a beam of light was directed, so that the reflected spot was focussed on a scale 70 cm. away. Pressure variations inside the capsule caused this spot of light to be drawn out into a band. A linear relation was found between the "deflection" and the pressure amplitude over the range required when the capsule was calibrated, whether statically by applying various small steady pressures (measured by a water manometer), or dynamically, by causing a little piston to oscillate at the appropriate frequency in the cylindrical portion of the capsule. The calibration curve for the capsule is shown in Fig. 6.

Table II compares the displacement amplitude at the mouth, with the pressure

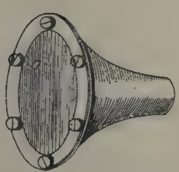

FIG. 5.—MANOMETRIC CAPSULE.

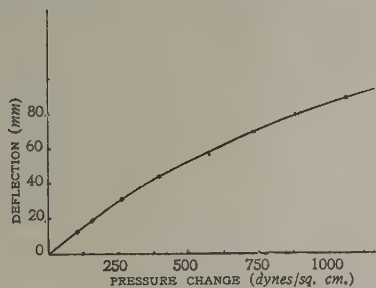


FIG. 6.—CALIBRATION CURVE OF MANOMETRIC CAPSULE.

amplitude in the interior for the resonator of frequency 126, at different amplitudes of the exciting fork.

TABLE II.

| Fork amp. cm. | Amp. at mouth. $a_0$ cm. | Press. amp. $P$ at rear (atmos.). | $a_0/P$. |
|---|---|---|---|
| 0·05 | 0·035 | 0·00011 | 320 |
| 0·08 | 0·055 | 0·00017 | 335 |
| 0·12 | 0·075 | 0·00030 | 250 |
| 0·20 | 0·125 | 0·00042 | 300 |
| 0.40 | 0·20 | 0·00075 | 270 |
| | | | Mean 295 |

whence the impedance of the orifice

$$Z_0 = P/(\pi r^2 \omega a_0) = 0{\cdot}31$$

assuming that $P$, the excess pressure amplitude, is constant throughout the interior. The theoretical impedance of this orifice is $\rho\omega/\kappa = 0{\cdot}25$, if $\kappa$ be put equal to the diameter.

Finally, a traverse was made through a Helmholtz double resonator. In this type a resonator has an orifice in the interior leading to another resonator. If the outer neck has conductivity $\kappa_1$, and the inner $\kappa_2$, the outer and inner reservoirs having capacities $v_1$ and $v_2$ respectively, then the lumped impedance of the inner orifice and vessel must be considered as in parallel with the impedance of the outer

orifice. To find the total impedance of these branches we add the reciprocals of their individual impedances and invert, thus:

Inner orifice and vessel:

$$Z_2=\varrho\left(\frac{\omega}{\kappa_2}+\frac{c^2}{v_2\omega}\right)$$

This, in parallel with outer orifice:

$$\frac{1}{Z_3}=\frac{1}{\varrho}\left(\frac{\kappa_1}{\omega}+\frac{\omega v_2\kappa_2}{\omega^2 v_2+\kappa_2 c^2}\right)$$

Outer vessel:

$$Z_4=-\frac{c^2\varrho}{v_1\omega}$$

To find the resonant frequency we put $Z_3+Z_4=0$.

$$\frac{\omega^3 v_2+\kappa_2\omega c^2}{\omega^2\kappa_1 v_2+\kappa_1\kappa_2 c^2+\omega^2\kappa_2 v_2}=\frac{c^2}{v_1\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

or

$$\omega^4+\omega^2c^2\left(\frac{\kappa_2}{v_2}+\frac{\kappa_1+\kappa_2}{v_1}\right)+\frac{\kappa_1\kappa_2}{v_1v_2}c^4=0$$

which is the equation obtained by Rayleigh,* using the "classical" methods.

The double resonator used in this experiment had an outer orifice of 3·2 cm. diameter ($=\kappa_1$), and an outer vessel of $v_1$=2,000 c.c. The inner neck was 2 cm. diameter and 2·65 cm. long ($\kappa_2$=1·2 cm.). The above equation has two roots, of which the smaller corresponds nearly to 143, the lower frequency to which the resonator responded. A traverse with the anemometer is given in Fig. 7.

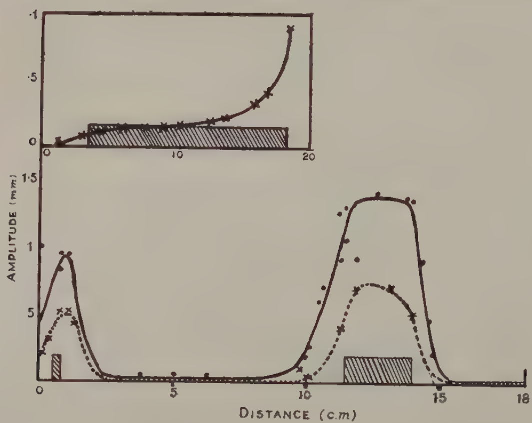


FIG. 7.—AMPLITUDE IN DOUBLE RESONATOR. INSET: DECAY OF AMPLITUDE IN LONG NECK.

Note the increased amplitude in the inner neck over the outer, in spite of the former's greater impedance; this is in accordance with a formula of Rayleigh—i.e.,

$$\dot{X}_1/\kappa_1+X_2/\kappa_2=0.$$

When very long necks were used, a notable decay of amplitude through the neck was noticed. This is shown (as inset to Fig. 7) for a neck 15 cm. long.

* "Sound," §310; see also Phil. Mag., 36, 231, 1918; and Paris, Phil. Mag., 2, 756, 1926. The latter author has worked out the theory of Boys' resonator, in which the outer resonator is replaced by a stopped pipe, on those lines. We could work out this case on impedance principles by substituting $c/S \tan kl$ for $c^2/v_1\omega$ in equation (7).

## III. Relation between Pressure in Single Resonator and Impedance of Mouth.

Applying equation (3) to the single resonator, if $p_0$ is the maximum value of pressure outside the mouth (due to the tuning fork), and $p_1$ that just inside, we have

$$p_1 - p_0 = Z_0 \dot{X} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

At the back of the resonator, where the pressure maximum is $p_2$ (this may be different from $p_1$), the current $\dot{X}$ is, of course, *nil*, and the impedance of the barrier is infinity. We have already tested equation (8) for the pure Helmholtz resonator. It was thought of interest to extend this part of the investigation farther by commencing with a cylindrical stopped pipe and applying the manometric membrane to a point close to the mouth, and then gradually restricting the mouth, taking corresponding readings of $p_1 - p_0$ and $\dot{\xi}$ in the mouth. The amplitude of the maintaining fork ($n$=220) was kept constant throughout the investigation, so that $p_0$ was preserved constant as far as possible, and the length of the cylindrical portion of the cavity (6 cm. diameter) was readjusted to best resonance with the fork each time. To this end, this part was made of two telescoping tubes. The metal plates with holes previously used, served as orifices, and some additional necks formed of 2·3 cm. diameter tubes of various lengths were also tested. Table III shows the results.

Table III.

| Orifice radius, cm. | $\kappa$ | $\frac{\varrho\omega}{\kappa}$ | $\dot{\xi}$ | $\dot{X}$ | $P\left(\frac{\text{dynes}}{\text{sq. cm.}}\right)$ | $P/\dot{X}$ |
|---|---|---|---|---|---|---|
| 0·75 ... ... ... | 1·5 | 1·1 | 120 | 210 | 250 | 1·2 |
| 0·825 ... ... | 1·65 | 0·96 | 27 | 57·5 | 85 | 1·5 |
| 1·0 ... ... ... | 2·0 | 0·8 | 27 | 85 | 85 | 1·0 |
| 1·25 ... ... ... | 2·5 | 0·64 | 18 | 90 | 85 | 0·95 |
| 1·5 ... ... ... | 3·0 | 0·53 | 22 | 155 | 70 | 0·45 |
| 2·3 ... ... ... | 4·6 | 0·35 | 20 | 83 | 40 | 0·48 |
| Neck, 2·3 cm. diam. | | | | | | |
| 4·1 long ... ... | 0·73 | 2·2 | 6 | 25 | 55 | 2·2 |
| 3·3 long ... ... | 0·85 | 1·9 | 8 | 33 | 44 | 1·3 |
| 2·25 long ... ... | 1·08 | 1·5 | 9 | 36 | 50 | 1·4 |

The numbers in the third column are the theoretical values of the impedance based on the usual values of the conductivities. In the last column are the corresponding values, based on the measured velocity and excess pressure amplitude at the mouth.

## IV. The Impedance of an Orifice.

In the simple theory of acoustic impedance viscous drag is neglected. The velocity over the cross-section perpendicular to the direction of vibration is supposed to be uniform, so that the inertance of an orifice becomes simply the mass of vibrating air in the neck. One means of calculating the true impedance is given by the figures in the last table, but it is better to attempt to calculate the impedance from an experimental determination of the actual mean velocity configuration across such an orifice, when the air is vibrating under simple harmonic impressed displacement.

Considering a lamina, $\delta x$ thick, in the plane of a circular orifice, the equation of motion is

$$\varrho A\ddot{\xi}\delta x+2\pi r\tau(1+i)\delta x=A\delta p$$

where $A$ is the cross-sectional area, $r$ the radius of the circle, $\tau$ the frictional force on unit area of wall, or simply the " skin friction, and $\delta p$ the pressure difference over the length $\delta x$ of the orifice.

Integrating over the length $l$ of the walls of the orifice in the direction of motion (=thickness of plate), and putting $\ddot{\xi}=i\omega\dot{\xi}$

$$2\pi r\tau l+i(\varrho\omega Al\dot{\xi}+2\pi r\tau l)=A\delta p \quad \ldots \quad (9)$$

In the type of motion here considered the régime is turbulent, and the idea is now generally accepted that, in such motion, the influence of viscosity may be considered as being confined to a thin layer in contact with the solid surface—the " boundary layer "—if the coefficient of friction $\mu$ is small. For simplicity we will consider that the velocity at any instant is uniform $(=\dot{\xi})$ over the major part of the orifice, but drops rapidly and with constant gradient to 0 at the boundary, in a small distance $\Delta$. This is the so-called " kinked " velocity profile discussed by Rayleigh* and by Prandtl.† Put

$$\tau=\mu\frac{\delta\dot{\xi}}{\delta r}=\frac{\mu\dot{\xi}}{\Delta}$$

where $\Delta$ is the thickness of boundary layer. Then

$$\frac{2\pi r\mu l}{A\Delta}+i\left(\varrho\omega l+\frac{2\pi r\mu l}{A\Delta}\right)=A\frac{\delta p}{\dot{X}}$$

Divide by $A$, and we have, by definition, the impedance of the orifice

$$Z_0=\frac{2\pi rl}{A^2}\cdot\frac{\mu}{\Delta}+i\left(\frac{\varrho\omega}{\kappa}+\frac{2\pi rl}{A^2}\cdot\frac{\mu}{\Delta}\right) \quad \ldots \quad (10)$$

where $\kappa$ has been put for the conductivity in the second term. The first term here is a pure resistance, but the same term is also involved in the reactance component.

Now, for simple harmonic motion it can be shown that $\Delta$ is proportional to $(\nu/n)^{\frac{1}{2}}$.‡ In fact, this quantity has the dimensions of a length, and we may consider the factor of proportionality $\varepsilon^2$ (which involves the nature and smoothness of the surface, among other things) as latent in the coefficient of viscosity, and we may put $\nu'=\varepsilon^2\nu$, so that

$$\Delta\equiv\sqrt{\frac{\nu'}{n}}=\sqrt{\frac{2\pi\nu'}{\omega}}\ :\ \nu=\frac{\mu}{\rho}.$$

The thickness of the boundary layer diminishes as the frequency increases, and the

* Lond. Math. Soc. Proc., 19, 67 (1887).
† Phys. Zeits., 23, 19 (1922).
‡ *Cf.*, Proc. Roy. Soc., 112, 536 (1926).

latter factor is, in fact, involved in all the terms of (10). Equation (10) becomes, with this substitution

$$Z_0=\frac{\sqrt{2\nu'\omega\pi}}{A^2}l\varrho r+i\varrho\left(\frac{\omega}{\kappa}+\frac{\sqrt{2\nu'\omega\pi}}{A^2}lr\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)^*$$

From an estimate of $\Delta$ from experimental values of the velocity at different points across the orifice, and the other known dimensions of the orifice, (10) serves to calculate the impedance when viscosity is included. The viscous terms give, in fact, the value of the resistance $F$, referred to in the first section.

The apparatus for testing the velocity in alternating flow through an orifice consisted of a reciprocating engine with long crank and adjustable stroke oscillating a light piston in one end of a cylindrical brass tube (internal diameter 6 cm.), which it just cleared. A vertical hot-wire anemometer was traversed horizontally across the section near the other end, either across the orifice formed by the mouth of the tube itself, or a hole in a plate soldered to the mouth. As the hot wire was necessarily a straight one, the sides of the orifice were distorted from the circular shape to form two opposite flats of 2 cm. long. A source of error arises from the use of such an anemometer, unfortunately, just over the region where the velocity needs to be known; this error is due to the cooling of the wire when close to the brass solid, by conduction across the intervening thin layer of air. For this conduction loss there is available a correction which is based on some experiments made in another connection by Dr. N. A. V. Piercy and the writer. The theory and application of this correction is discussed elsewhere,† but it suffices to say that in order to determine this correction the hot wire under definite conditions of heating current, temperature, etc., was whirled round at known speeds and at known small distances from a large cylindrical brass surface. From the calibration curve of the anemometer, obtained in the usual way, the apparent velocities corresponding to the measured values of the resistance were found, and these subtracted from the true velocities gave the correction necessary at this particular distance from the surface. In the present experiments a sample of the same nickel wire was used at the same excess temperature, and the correction so obtained was applied to the apparent velocities when the anemometer was close to the brass surface. The resulting velocities are still approximate, but are the best that could be obtained under the difficult circumstances. The frequency was kept constant by stroboscopic means.

Table IV gives the figures, and Figs. 8 and 9 give the gradients, for a number of frequencies. Using the "hole-in-plate" orifice, a resonator was formed to respond to the tuning fork previously used ($n=220$), and a traverse across the orifice was made with the hot-wire anemometer, while the resonator was excited by means of the tuning-fork.

This result is included in Fig. 8, but the rest of the results were obtained by means of the oscillating engine at low frequencies and rather large amplitudes, as the hot wire was more sensitive and the boundary layer thicker under these con-

* In this form the expression for the impedance is practically the same as that deduced by Stewart, Phys. Rev., 27, 489 (1926), save that he puts the unchanged coefficient $\nu$, which is tantamount to putting $\Delta\equiv(\nu/n)^{\frac{1}{2}}$.

† In a paper submitted to the Aeronautical Research Council

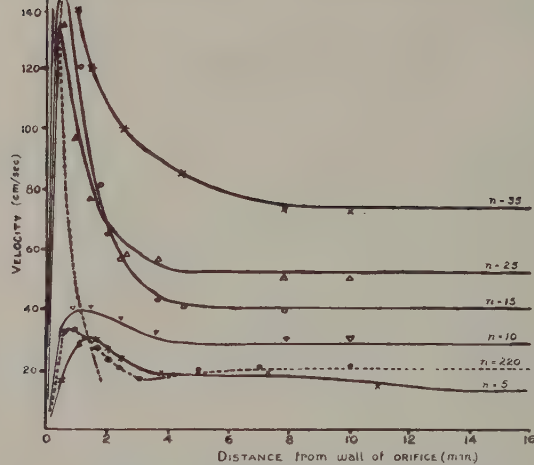


FIG. 8.—VELOCITY ACROSS ORIFICE (1·75 CM. RADIUS).

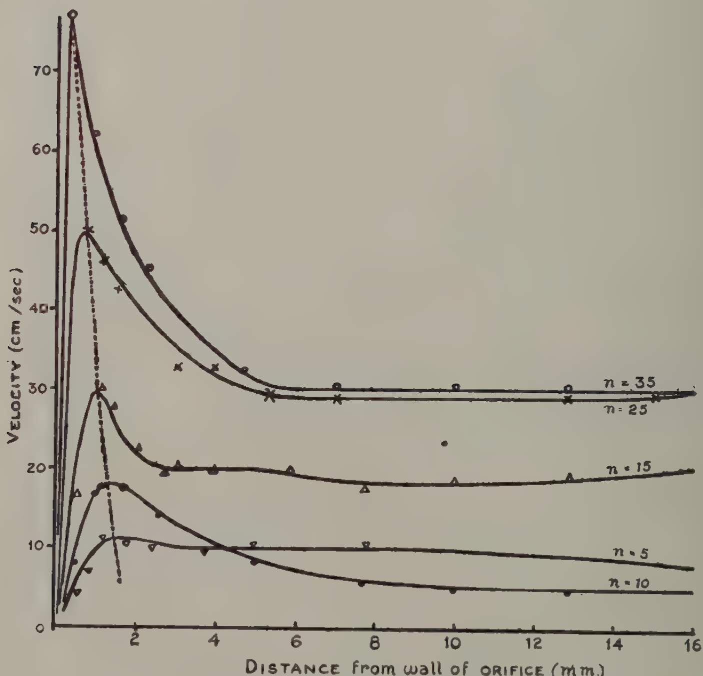


FIG. 9.—VELOCITY ACROSS ORIFICE (3 CM. RADIUS).

ditions. The type of curve obtained seemed to be the same for both sets of experiments, whether the resonator or the engine was used.

TABLE IV.

(Fig. 8) $r=1{\cdot}75$ cm., $l$(=thickness of plate)=2 mm.

| $n$ | $\Delta$ | $V_{max.}/\Delta$ | $\Delta\sqrt{n}$ | $F(\times 10^5)$ |
|---|---|---|---|---|
| 5 | 0·14 | 180 | 0·31 | 3·0 |
| 10 | 0·11 | 350 | 0·35 | 3·8 |
| 15 | 0·075 | 2150 | 0·29 | 5·6 |
| 25 | 0·06 | 2450 | 0·30 | 7·5 |

(Fig. 9) $r=3$ cm.

| $n$ | $\Delta$ | $V_{max.}/\Delta$ | $\Delta\sqrt{n}$ |
|---|---|---|---|
| 5 | 0·13 | 85 | 0·29 |
| 15 | 0·10 | 290 | 0·38 |
| 25 | 0·075 | 715 | 0·35 |
| 35 | 0·05 | 1600 | 0·30 |

The table annexed to the figures shows that $\Delta \propto n^{-\frac{1}{2}}$, as predicted, reckoning $\Delta$ as the distance of the peak velocity from the boundary. But there is another curious result : $\dot{\xi}$ does not reach a nearly constant value at the edge of the boundary layer, but there is a drop of velocity before a uniform value is reached. Whereas the boundary layer gets thinner, this region of high velocity broadens out with increasing frequency. In calculating $Z$, therefore, it seems necessary to take an average value of $\dot{\xi}$ outside the boundary layer, which value will be somewhat greater than the value at the centre. With this modification, and with a value of $l$, the effective length of the orifice, the value of the frictional resistance

$$F=\frac{2\pi rl}{A^2}\frac{\mu}{\Delta}$$

has been calculated (Table IV). It is naturally small compared with the inertance of the air in a wide orifice, but becomes of importance when the cross-section of the orifice is attenuated. Referring back to Fig. 1, the frictional resistance seems to diminish the admittance (= the reciprocal of the impedance) when the conductivity of the orifice is small causing the experimental curves to rise above the theoretical (viscosity neglected) as the origin is approached. In other words, the length assigned to a measured value of $r$ is really appropriate to a lesser $r$ when viscous effects are considered.

The crowding of the motion into the outer rings of the orifice at higher frequencies reminds one of the " skin-effect " in A.C. technology. and is probably an inertia effect. The effective inertia of the air in these parts of the mouth is less in virtue of the fact that, in vibration, it can push the added mass of the outer air " round the corner," along the flange, whereas the central layers are unable to do this, so that their effective inertia is greater. It may be remarked in parenthesis that no such " skin-effect " was observed when " one-way " turbulent flow was produced in the orifice.

It seems likely that, for a given area of orifice, the conductivity may be increased

by making the orifice in the form of an annular ring instead of a circle, since the vibrating air tends to crowd into the annulus. Two orifices of the pattern shown in Fig. 10 were constructed, and also two circular tube orifices of equal cross-sectional area, and applied to the cylindrical resonance tubes of Tables I and II, with the following results.

TABLE V.

Resonant lengths for tube, 4·3 cm. diameter.

| Cross-section, sq. cms. | | $n=$ 500 | $n=$ 350 | $n=$ 250 | $n=$ 220 | $n=$ 100 |
|---|---|---|---|---|---|---|
| 3·1 | Circle ... | 11·0 | 18·3 | 28·0 | 33·3 | 42·5 |
| | Ring ... | 12·6 | 19·5 | 29·5 | 34·5 | 44·0 |
| 3·55 | Circle ... | 13·2 | 21·0 | 30·0 | 35·5 | 44·6 |
| | Ring ... | 14·2 | 21·7 | 30·8 | 36·4 | 46·5 |

It appears from this table that the resonant lengths for the annular orifice are somewhat longer than those for the circular orifices of equal size, thus confirming the prediction of a less impedance for the annular orifice. This fact might be made use of in constructing hot-wire microphones, as the sensitivity of response of the hot wire should be greater if placed near the circumference of a hole, or in an annular space like that of Fig. 10.

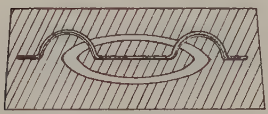

FIG. 10.—ANNULAR ORIFICE FOR RESONATOR.

The peculiar case of alternating airflow in a tube or orifice has evidently other interests apart from the present sound problems, and will be treated in further experiments.

## CONCLUSION.

It is hoped that this paper demonstrates the advantage of impedance methods in application to pipes and resonators both in theory and practice. A number of physicists have been adapting the concept of acoustic impedance in connection with the construction of acoustic filters; in particular, Stewart* has tested the efficiency of such filters against theory, and has found certain deviations. The present Paper describes attempts at a *direct* measurement of impedance, and the non-uniform flow across an orifice under certain circumstances which is herein demonstrated may explain the deviations from theory which are observed when such orifices are used in filters. The fact, which several workers on end-corrections have noted†—and which is confirmed by Figs. 1 and 2—that the value of the end-correction of a pipe varies a little with frequency, may in part be due to the same cause, since the extent of the non-uniformity in flow across the section of an orifice when the air is vibrating in it is a function of the frequency of such vibration.

This work was carried out in the Carey Foster Laboratory at University College, London, and I am indebted to Prof. A. W. Porter, D.Sc., F.R.S., for facilities to carry it out, and for encouragement during its course.

* Phys. Rev., 27, 818 (1926).

† *Cf., e.g.*, Anderson and Ostensen, *loc. cit.*

## DISCUSSION.

Dr. A. H. DAVIS suggested that a more detailed analogy between the " annular effect " (viz., the increase in particle velocity which occurs close to the edges of the orifice of a vibrating resonator) and the skin effect obtained with high frequency electric currents might conceivably be worked out by comparing the vibration of air at small amplitudes (the air being treated as an incompressible fluid) with high-frequency current flow in a mass of copper divided into two parts by an embedded slab of insulation with one circular conducting portion. The author had shown the value of the impedance method as applied to acoustics, but his derivation of equation (7) really depended on Lord Rayleigh's method, since it took the value of the " conductivity " for granted.

Mr. A. G. WARREN said that the " annular effect " was analogous to an effect which occurs in the moving-coil loud-speaker. He had calculated the pressure distribution over a disc vibrating harmonically in an infinite plane wall, not by the Rayleigh-Maxwell method of finding the common potential, but by integrating first circumferentially and then radially, the disc being supposed complete throughout. This method, though laborious, gives the radial distribution of pressure. The pressure has components in phase respectively with the velocity and with the acceleration, and these components are uniform over the disc at low frequencies. As the frequency increases the pressure at the edge first falls, particularly in the case of the component in phase with the acceleration, and then reverses, and at high frequencies there is a Bessel wave distribution of pressure over the disc. A similar method might be applied to the study of the Author's " annular effect " ; and since in the case of the loud-speaker, where the velocity is uniform over the disc, the pressure diminishes at the edge, in the case of the resonator, where the pressure is more or less uniform, it is to be expected that the velocity will increase towards the edge. This conclusion is qualitatively consistent with the author's results, but the latter give a surprisingly large increase in the velocity near the edge of the orifice.

Dr. J. H. VINCENT said it would be interesting to know the absolute amplitude of movement in the orifice of the resonator. Would it not be possible to observe the movements of smoke particles in that region ?

Dr. W. H. ECCLES expressed interest in the fact that whereas alternating current theory had originally been based on Lord Rayleigh's acoustical conception of the complex exponential, the conception of impedance which had been added by electrical theory, was now being transplanted back into acoustics. What were the relative sensibilities of the hot-wire anemometer and the human ear ?

Dr. E. T. PARIS (communicated) : Dr. Richardson's formula (6) for the frequency of a stopped pipe was given by Rayleigh in 1871, though the fact that it could be used for pipes of lengths small compared with the wave-length of the fundamental tone (that is, cylindrical Helmholtz resonators) was not actually stated. One advantage of a formula of the type (6) over the usual frequency-formula (2) for a Helmholtz resonator is that the former gives the frequencies of overtones, whereas the latter gives only the frequency of the fundamental. It is to be noted, however, that (6) is strictly applicable only to the case of resonators with cylindrical bodies.

A more general formula for resonators with cylindrical parts was published by Mr. J. A. Aldis in 1924 (" Nature," Vol. 114, p. 309), who claimed to have used it as long ago as 1867. It refers to resonators (or organ-pipes) made with two pipes of unequal diameter joined together with a flange from the outer edge of the smaller pipe to the inner edge of the larger stopped pipe (making a " bottle-pipe "). The formula for the resonance frequencies is $\tan kL_1 \tan kL_2 = \sigma_1/\sigma_2$, where $L_1$ and $L_2$ are the lengths and $\sigma_1$, $\sigma_2$ the cross-sectional areas of the smaller and larger pipes respectively, and $k=2\pi/\lambda$. As I pointed out at the time (" Nature," Vol. 114, p. 465), by taking $L_1$ and $L_2$ both small compared with $\lambda$ Rayleigh's formula for the frequency of a Helmholtz resonator can be obtained as a limiting case. Also if $L_1$, but not $L_2$, is supposed small compared with $\lambda$ we get the formula applicable to a stopped pipe with restricted mouth of conductance $c=\sigma_1/L_1$ ; if $L_2$, but not $L_1$, is supposed small, we get the formula given by Rayleigh for a long tube connected with a reservoir of small volume.

Mr. E. J. IRONS (communicated) : For some time past I have been engaged on the problem of the vibration of air in a partially stopped tube, and I take this opportunity of offering my congratulations to the author for his contribution to the subject.

Whereas Dr. Richardson has made a frontal attack on the orifice with a tuning fork, in my work—chiefly concerned with the determination of end corrections—I have deemed it advisable,

for reasons already stated (Phil. Mag. 5, 580, 1928), to adopt the method of approaching the orifice from the rear. Results obtained by a Kundt's tube method have already been described (Phil. Mag., loc. cit.), and a new apparatus, consisting essentially of a hot wire detector of nodes and antinodes, in a tube having at one end an orifice of known diameter, and at the other a valve maintained source of vibration, is at present in use.

I note that Dr. Richardson has replaced the method for determining the resistance of the hot wire which he described in his Royal Society Paper by a form of Kelvin bridge. I suggest that experimental procedure would be simplified by including the balancing and sliding resistances of his first arrangement in the same arm as the hot wire.

AUTHOR'S reply: In considering the analogy between the acoustical and the electrical skin-effect, I have found it difficult to define an acoustical analogue of the inductance to which the electrical skin-effect is due. It is difficult to think of any experimental method for measuring oscillatory pressure; the Pitot tube would show nothing. I have observed the distribution of velocity in the case of direct flow of air (analogous to direct current), but no annular accumulation of flow was indicated by the hot-wire. I am grateful to Dr. Vincent for his suggestion to observe smoke particles in the orifice. Table III shows the amplitude ($=\xi/2\pi n$) in the orifice of a resonator to be of the order of a millimetre.

The impedance method does not, of course, supersede the classical method; it is another aspect of the same physical phenomena, but might be better suited for teaching purposes, as it avoids the use of the conception of velocity potential. In calculating Tables I, II and III I have assumed Rayleigh's formula for the conductivity of a circular hole (=the diameter), but I do not think I have taken the conductivity for granted in equation (7). In the course of deriving equations (6) and (7) I have merely put $k$ as a symbol for $A/l$. I thank Dr. Paris for the reference to his letter in Nature, which I had overlooked. Equation (6) is implicit in all the formulæ for end-corrections (Rayleigh, Helmholtz, etc.), but has not, I believe, been correlated with the impedance of the end of the pipe. Mr. Aldis mentions tests on his double pipes, but there were, I think, no actual data on partially stopped pipes, except those of Mr. Irons, published in March of this year, which were obtained with another object. I found the Kelvin bridge arrangement (due to L. V. King) better suited to measurement of velocities than the one I formerly used, in so far as change of zero due to room temperature merely shifts the whole set of readings up or down the bridge-wire by a constant amount.

In reply to Dr. Eccles, I understand that the hot-wire will detect sounds inaudible to the unassisted ear if the hot-wire be combined to a tuned resonator, as in Dr. Tucker's instrument.

# XXVIII.—THE FOCUS OF A GAS-FILLED X-RAY TUBE.

By R. E. Clay, *M.Sc.*, Davy Faraday Research Laboratory, Royal Institution.

(*Communicated by* Prof. Sir William Bragg, *K.B.E., F.R.S.*)

*Received March* 20, 1928.

Abstract.

The importance of obtaining a sharp focus in an X-ray tube used for crystal analysis is emphasised. Pinhole photographs of the focus obtained with various radii of curvature of the cathode and various distances from the anticathode are discussed and the conclusion drawn that with the tubes of the type considered, a radius of about 2 cms. and a distance of 3 or 4 cms. are the best conditions.

For crystal analysis it is very often necessary to confine the X-rays incident on the crystal to a fine pencil; this is usually effected by passing them through a narrow channel or through two pinholes some distance apart as indicated roughly in Fig. 1.

Fig. 1.

These two holes, which correspond to the collimator of an ordinary spectrometer define a cone which intersects the surface of the anticathode in a small area. In order to get the maximum efficiency from the X-ray tube, the focus where the X-rays are produced has to be within this area or preferably to cover it completely yet without being larger.

Any X-rays produced outside this area are wasted. This important fact is sometimes overlooked by X-ray workers with the result that much time is wasted in making long exposures.

In sealed-off X-ray tubes the focus is usually sufficiently small to prevent this loss, but for crystal analysis the ionic type of X-ray tube working in conjunction with a pump is very often used nowadays. In these tubes the size of the focus depends not only on the shape, size and relative position of the cathode and anticathode, but also upon the gas pressure in the tube.

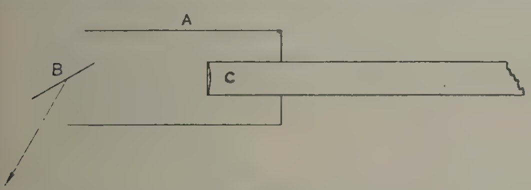


Fig. 2.

Very little systematic work on the modern type of gas-filled tube has been published, and it was therefore thought that a study of the conditions for maximum efficiency with regard to the size of the focus would be of interest.

The work to be described was carried out with one of the "standard" tubes in use in this laboratory which was developed by Shearer and is described by Bragg.* Fig. 2 shows the important parts; these are the cathode $C$—an aluminium rod of diameter 1·4 cms. encased in brass tube—the end of which is turned to a portion

* "X-Rays and Crystal Structure," pages 34 and 295.

of a sphere; the anode sleeve *A* which envelops the end of the cathode (the diameter of this was 4 cms.) and the anticathode *B*. *A* and *B* were connected to earth and *C* to the high potential terminal of the induction coil.

The end of *C* was turned in a lathe to a series of different radii—in the case of the larger radii (5 cms. and upward) by means of a pivoted tool, and in the case of the smaller radii by trial to a template.

The focus spot was photographed by means of a pinhole camera. A piece of lead with a fine hole in it was placed about 15 cms. from the tube and a photographic plate contained in a dark slide of which the front had been cut away and replaced with black paper was placed to receive an image of the focus spot. Any desired magnification can be obtained by altering the distance from pinhole to plate: in this work a magnification of 2 : 1 was used, that is to say, the plate was placed 30 cms. from the pinhole.

In most of the experiments the tube was run on an induction coil with a Wehnelt interrupter; it was connected to the pump throughout the experiments and the pressure was adjusted by a screw pinch-cock which controlled the rate at which air could leak in from the fore vacuum; the pressure in the fore vacuum was maintained constant in a similar way by keeping the backing pump running and allowing air to leak in from the atmosphere—in this way a stream of air is continuously circulated through the whole apparatus.

The pressure was measured with a McLeod gauge having a compression ratio of $10^4 : 1$.

A series of photographs was taken for various radii of curvature and various distances with a cathode of the standard diameter (1·4 cms.). Fig. 3 shows the results obtained.

The horizontal scale indicates the distance in cms. from the anticathode, while the vertical scale indicates the radius of curvature in cms. of the cathode surface. The actual focus in the X-ray tube was of $\frac{3}{4}$ the linear dimensions of the spots in the figure.

The smallest focus is obtained with the 1 cm. radius at a distance of 1·5 cms. but the tube was very unstable and the pressure adjustment very critical. The same is to be said of the next radius and the small distance at which again a good focus is obtained. In the case of a radius of 2 cms. the best focus is obtained over a range from 3 to 4 cms. from the anticathode, the tube is stable under these conditions and the pressure range over which a good focus is obtained is fairly wide. For larger radii the focus is larger whatever the distance and no improvement is made in the stability and general running of the tube.

A cathode having a radius of curvature of 2 cms. and placed 3·5 cms. from the anticathode appears, therefore, to give the most satisfactory results.

The gas pressure depends upon the distance; the figures in the right-hand column give in mms. of mercury the pressure for the distance at which the focus was a minimum. The current through the tube was of the order 1-2 milliamperes throughout and the alternative spark gap was usually about 12 mms. between spheres of 25 mms. diameter.

The effect of pressure variation is shown in Fig. 4 for two cathodes, one the standard, 1·4 cms. in diameter, and the other 2·4 cms. in diameter; the radius of curvature was 2 cms. in each case. A group of three photos was taken for each of the distances 3·0, 3·5, 4·0 cms., and in each group the pressure was adjusted to

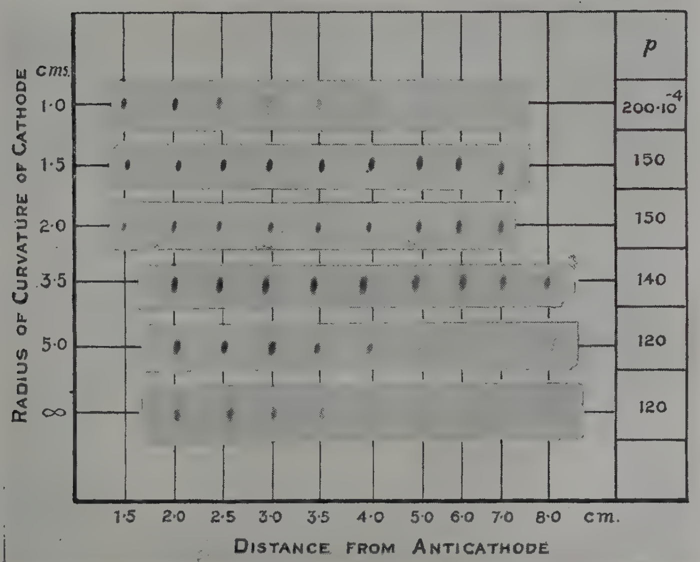


Fig. 3.

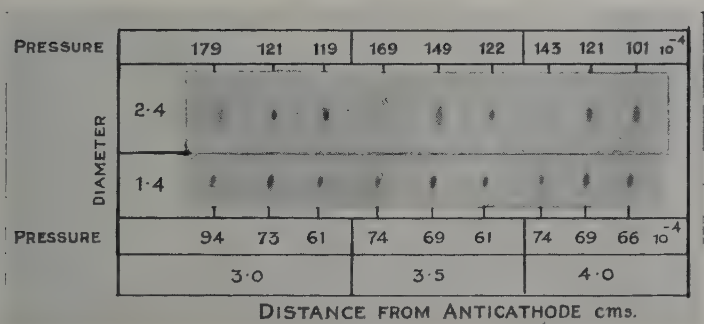


Fig. 4.

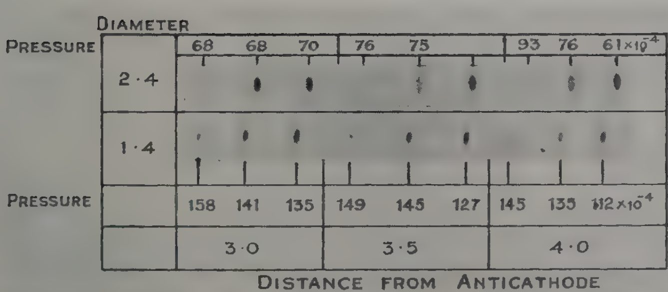


Fig. 5.

*To face page* 222]

three values—the maximum at which X-rays could be obtained, that at which the strongest beam was obtained and the minimum at which X-rays could be obtained.

The effects are most marked in the case of the larger cathode and it is probable that some radius other than 2 cms. would prove to be more suitable for a cathode of this diameter.

The behaviour of the tube is very much the same when it is run on a high-tension transformer from A.C. mains and the same conditions for the best focus hold ; the variation of focus spot with gas pressure is a little different owing, no doubt, to the voltage being a different function of the current. Fig. 5 is identical with Fig. 4 except that a transformer was used in place of the induction coil.

## DISCUSSION.

Major C. E. S. PHILLIPS referred to some experiments by Campbell Swinton on an early form of gas X-ray tube (with a carbon target) used in electro-therapy. In these experiments, and in the speaker's experience, the cathode stream always appeared to be hollow, so that the focus was surrounded by concentric rings. He was surprised that changes in gas-pressure should have so little effect. In his experience the length of the parallel part of the cathode stream varied considerably with pressure, and this circumstance would affect the size of the focus. Campbell Swinton's tubes were not highly evacuated, however. What was the pressure in the tubes to which the paper referred ?

Mr. A. G. TARRANT suggested that the shape of the beam under various conditions could be reconstructed from the cross-sections obtained by the author.

Dr. L. SIMONS said that as the author had arranged the photographic plate at right angles to the X-rays, whereas the target was oblique to the rays, the photographic spot differed in shape from the focus on the target. If the plate and target were equally inclined to the X-rays the true shape of the focus would be recorded.

Mr. A. G. WARREN : I would suggest that the Author might profitably include in the Paper an explanation of focussing. In recent years a number of curious ideas have been current, such as that the focus at normal voltage is appreciably affected by mutual repulsion of the constituents of the cathode stream. I suggest that the focus can be sufficiently explained by the direction and magnitude of the electron speed acquired in the neighbourhood of the cathode, and that subsequent deviations are small. The presence of gas is chiefly of importance in influencing the potential gradient at the surface of the cathode.

Dr. A. MULLER said that the object of the Paper was purely practical. It had been found that the effective exposure with the Shearer tube is very uncertain, but in the light of the present investigation into the best working conditions excellent results had been obtained.

Dr. W. H. ECCLES asked what was the composition of the residual gas in the Shearer tube.

The AUTHOR, in reply, pointed out several of his photographs which indicated a hollow structure in the cathode stream, and said that the focus, for various positions of the anticathode, could scarcely be regarded as a section of the same invariable cathode stream, since the field, and therefore the form of the cathode stream, would be influenced by the position of the anticathode. The object of the paper was to study the size of the focus as viewed from the crystal, but the distribution of intensity in the focus itself could be inferred by geometrical reasoning if required. The residual gas in the tube was atmospheric, though Dr. Müller had used tubes containing carbon dioxide.

# XXIX.—A NOTE ON THE ELASTIC IMPACT OF THE PIANOFORTE HAMMER.

By R. N. Ghosh.

ABSTRACT.

The present paper comprises a critical discussion of a previous paper by Mr. P. Das.

IN a recent paper* on the elastic impact of the pianoforte hammer, Mr. P. Das has brought out certain new features on theoretical grounds which deserve further consideration in the light of experimental work. Starting with Kaufmaun's case when the impact takes place near one end of the string, he obtains a formula for the pressure

$$P=\frac{VE}{c\,q}\left[\frac{1}{q}\left(e^{\xi ct}\cos\,\eta ct-e^{qct}\right)-\frac{1}{a\eta}\,e^{\xi ct}\sin\eta ct\right]\quad\ldots\ldots\ldots(1)$$

$V$ being the initial velocity of hammer.

Whatever be the value of the elasticity of the hammer, (1) shows that $P=0$ at $t=0$. On this basis Mr. Das questions the validity of the author's formula† for the pressure $P$

$$P=\frac{VT_0}{q}\,e^{-\frac{1}{2}kt}.\left[\frac{\sin qt}{a}+\frac{R}{c}\cos\,(qt+\varphi)\right].\quad\ldots\ldots\ldots(2)$$

which gives a finite value for the pressure at the beginning of impact. Formula (2) has been derived on the secure foundation of experimental results—viz., that the velocity of the striking point of the string differs little from that of the initial velocity of the hammer. This is shown by the numerous photographic records of the simultaneous motion of the hammer and the string taken both by the present author and Dr. W. G. George. The pressure is actually given by

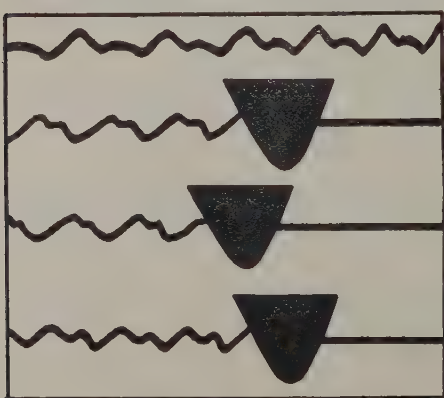

FIG. 1.

$$P=T_0\left[\frac{T_0}{a}+\frac{1}{c}\;\frac{dy_0}{dt}\right]$$

and hence, if $T_0=0$ and $dy_0/dt=V$, we get $p=V_0T_0/c$ at the beginning of impact. Mr. Das supposes $dy_0/dt=0$ initially. Herein lies the error (*see* Fig. 1).

It is not difficult to show that this pressure initially arises from the loss of momentum which is communicated to the string. Such a type of velocity imparted to a finite portion of the string will approximately produce a motion similar to that

* Proc. Phys. Soc., Vol. 40, p.29 (1927).

† Proc. Phys. Soc., Vol. 40, p. 31 (1927). $q$ in (1) and (2) has different meanings, original symbols being preserved.

of the bowed string. This is found to be the fact in many cases.* Equation (6) of Mr. Das seems to be in error—the factor $m_0$ occurs with $y''$ and not with $y'''$. The term in $y'''$ is small and of order $10^{-3}$ in comparison with others in the case of felt hammers contemplated by the author. As pointed out above, this term does not affect the initial value of the pressure.

Experimental observations on the pressure of impact have been made by Prof. Raman and Dr. W. George in the case of hard hammers. Their results definitely show that the pressure is finite at the beginning of impact: (1) shows, however, that, even if elasticity tends to large values, still pressure must always be zero initially; while (2) agrees with the experimental facts as the hammers become hard. Thus it is clear that (2) has wider practical application than (1). Probably in the case of light and very elastic hammers, like old tennis balls impinging upon the string, (1) may hold; but it remains an open question for experiment to decide.

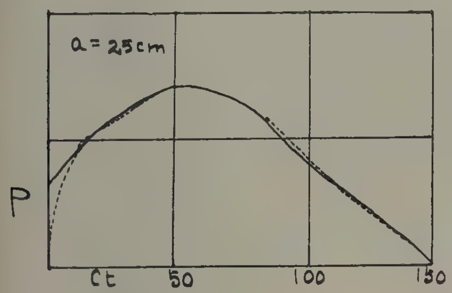

FIG. 2.

Fig. 2 shows the pressure variation during impact as calculated from author's formula (2), the dotted curve being that due to (1). The difference in the initial stages is, as pointed before, due to the assumption $\dot{y}=0$. Since $\dot{\zeta}=\dot{y}_0+\dot{u}$ where $\dot{\zeta}$=velocity of hammer, $u$=compression, it is evident that as the hammer increases in hardness $\dot{y}_0$ will certainly approach $\dot{\zeta}$. And in practice it has been found that in the case of felt hammers, as mentioned before, there is very little difference between $\dot{y}_0$ and $\dot{\zeta}_0$ initially; therefore (2) applies directly to the case of pianoforte hammers.

* Proc. Roy. Soc., Vol. 108, p. 294, plate 9. The full equation of motion of the string is given in Phys. Rev., Vol. 24, p. 457 (1924).

## DEMONSTRATION OF "DIE RASTERMETHODE" (DETECTION OF SPHERICAL ABERRATION BY MEANS OF SHADOWS IN ASTIGMATIC BEAMS).

By J. E. Calthrop, *M.A.*, *M.Sc.*

THIS method, which has been applied recently by Jentzsch (Physik. Zeits., No. 750, pp. 66-72, January, 1928) and Schulz (Ann. d. Physik., p. 66, XIX, 1928) to the measurement of the spherical aberration of lenses, consists in forming a real image of a small source, such as pointolite or carbon-arc lamp, and inserting a grating in the neighbourhood of the image. The effect is viewed on a distant screen, and owing to the fact that the rays come from a caustic curve they produce a shadow which is a distorted image of the grating. From measurements of the distortion it is possible to find the spherical aberration of the lens.

The effects for a single line object were known to Prof. Silvanus Thompson in 1903, and a theoretical treatment was also given by Bennett (Proc. Phys. Soc., p. 205, XIX, 1904) in a Paper entitled "Notes on Non-homocentric Pencils and Shadows Produced in Them. I: The Standard Astigmatic Pencil."

On the present occasion the curious shadows produced by lines, gratings and gauzes were demonstrated, and the differences between a corrected and uncorrected lens were illustrated.

## DEMONSTRATION OF THE BLOWING OF SELENIUM BUBBLES.

By Major C. E. S. Phillips, *F.R.S.E.*

THE object in view is to obtain thin films of selenium for observation under a microscope. The selenium is heated in a beaker, and a blow-pipe having a slightly flared end is warmed to the same temperature by immersion in it. The blow-pipe is connected through a length of rubber tubing to the mouth of the operator, who withdraws the blow-pipe from the melt, and, testing the consistency of the adhering selenium by alternate air pressure and suction, selects the right moment for blowing the bubble. The method is uncertain, but with practice good results can be obtained, and two large sausage-shaped bubbles were blown in the course of the demonstration. Pieces of the selenium film thus produced have the property of seizing or cold-welding when brought into quite light contact with one another.

## DEMONSTRATION OF A PORTABLE ELECTRIC HARMONIC ANALYSER, SHOWING THE MEASUREMENT OF HARMONICS IN VOLTAGE AND CURRENT WAVES.

By R. Thornton Coe, *B.A.*, *M.Sc.Tech.*, *A.M.I.E.E.*

THE instrument is used by the British Thomson-Houston Co., Ltd., for measuring small harmonics in voltage or current waves. The determination of each harmonic is made separately from steady readings on two instruments, and only takes a few minutes. The amplitudes of small harmonics in voltage waves are found correct to at least 1/20th of 1 per cent. of the fundamental.

The method embodies the mathematical principle of obtaining the $n$th harmonic in a wave by multiplying by sin $npt$ and integrating over a complete period.

The current to be analysed is passed through the moving coil of a dynamometer, while through the fixed coil is passed a sinusoidal "analysing current" of exactly the frequency of the harmonic under investigation; this is obtained by the use of a tuned circuit, valves and a synchronously driven contact disc. The phase of the analysing current is changed to give a maximum dynamometer reading $D_{max.}$, where

$$D_{\text{max.}} = kI_a I_n.$$

The analysing current $I_a$ is measured on a thermal ammeter, and since $k$, the dynamometer D.C. calibration constant, is known, the required harmonic of current $I_n$ can readily be calculated. A large current is analysed by connecting a suitable shunt across the dynamometer moving coil, while a voltage is analysed by connecting a fixed condenser in series. The apparatus was shown in operation upon the voltage wave-form of current taken by a small auto-transformer supplying the synchronous motor of the analyser. The most prominent harmonics in the voltage wave were the 3rd, 5th and 7th, and also the 17th and 19th. The former (of amounts 2·1 per cent., 3·9 per cent. and 1·25 per cent. respectively on a fundamental of 112 volts) were due to the somewhat rectangular wave-form of flux distribution round the armature, while the latter (of amounts 3·0 per cent. and 1·5 per cent. respectively) were due to the ripple of flux caused by the slots in which the armature conductors are placed.

The transformer current wave showed large 3rd and 5th harmonics of amount 13·1 per cent. and 4·5 per cent. respectively on a fundamental of 5·2 amps. These large harmonics were due to the well-known distortion of the magnetising current wave of a transformer when the maximum flux-density is sufficient to saturate the iron core.

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